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Die Typen der linearen Complexe rationaler Curven im R_r.

Von S. Kantor.

In der Theorie der endlichen discontinuirlichen Gruppen, welche ich für die Ebene in meinem Buche* feststellte, hat sich mir nach mehrfachen Versuchen die Ueberzeugung befestigt, dass auch für den R_r jene Theorie nicht eigentlich auf invariante Functionenkörper von $r+1$ Variabeln—die invarianten M_{r-1}-Systeme—zu begründen ist, sondern dass vielmehr die invarianten Curvencomplexe zum Angelpunkte für die Entdeckung der Aequivalenztheoreme werden. Bewogen durch diese Ueberzeugung hatte ich bereits in der ersten Hälfte des Jahres 1896 einen Abriss jener Theorie auf dem Fundamente der Curvencomplexe skizzirt. Es erweist sich zu ihrem vollständigen Abschlusse als eine nothwendige Präliminararbeit, Theoreme zu finden, welche für die rationalen und elliptischen Curven im R_r dasselbe leisten, wie jene, welche man in der Ebene theils seit langer Zeit, theils durch meine 1899 (Monatshefte) veröffentlichten Arbeiten kennt. Und eben im R_r zeigt sich die Nothwendigkeit dieser letzteren neuen Theoreme, welche alles Irrationale ausschliessen.†

Wie im R_2 sind auch im R_r zwei Reihen von Theoremen zu bilden. Die eine sieht alles als gegeben an und hat also allgemein nur einen numerischen Werth; die zweite reicht in allen Fällen, auch wenn die Basiselemente nur gruppenweise rational bekannt sind, aus, wobei also der natürliche Rationalitätsbereich der geometrischen Data nicht überschritten zu werden braucht. Die Superiorität der zweiten Reihe liegt auch hier darin, dass sie eigentlich vor die erste Reihe zu treten hat. Denn diese kann aus jener gefolgert werden, während

*Mayer u. Müller, 1895.

†Denn in den Fällen, wo die Theoreme aus der Ebene zur Anwendung im R_r gelangen, wird, was in der Ebene discrete Punktgruppe war, ein Weber-Dedekind'sches Polygon auf einer algebraischen Varietät und verliert die für die Betheiligung an einem Fundamentalsysteme nothwendige Bestimmbarkeit.

sie selbst derzeit wohl überhaupt eine arithmetische Herleitung nicht gestattet, sobald $r > 2$. Ich spreche diese Meinung aus* obzwar ich selbst in Acta Math. XXI. p. 1–78 für eine sehr ausgedehnte Classe von Systemen die arithmetischen Theoreme auf dem Wege der Relationen für die Singularitäten entwickelte.

Aus dem Folgenden sei das durchgehends zur Geltung kommende Princip der Transversalmannigfaltigkeiten hervorgehoben, das die Theorie im R_r ohne die Herbeiziehung noch höherer Räume in zufriedenstellender Weise aufzubauen gestattet. Diese Methode enthält das früher (Acta Math. XIX. und C. R. 1885) in Anwendung gebrachte Princip der Verminderung der ϕ als speciellen Fall in sich.

In §4 habe ich eine von Domenico Montesano, einem der besten lebenden italienischen Geometer, als angeblich typisch durchgeführte Eintheilung der linearen ∞^2-Systeme von Kegelschnitten im R_3 als in diesem Sinne falsch nachgewiesen.

In §5 gebe ich die Verallgemeinerung eines vielgenannten Picard'schen Satzes† über M_2 im R_3 auf M_{r-1} im R_r.

§1.—*Allgemeines über lineare Complexe von Curven im R_r.*

1. Ein linearer ∞^{r-1}-Complex von Curven C_k im R_r ist ein solcher, von dem durch jeden Punkt des R_r nur eine der C_k geht.

Lemma.‡ *Jede Involution von ∞^r k-punktigen Gruppen im R_r ist rational.*

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* Der Grund für diesen Ausspruch ist eben in der Unhandlichkeit der Nöther'schen Postulationsformeln zu suchen, auf die ich bereits Acta Math. XXI. hingewiesen habe.

† Cr. J. Bd. 100.

‡ Ich habe diesen Satz schon in Acta Math. vol. XIX ausgesprochen. Ich würde es aber gar nicht nothwendig haben, den Satz hier zu beweisen, wenn ich nicht meine Theoreme durchwegs allgemein für "alle Complexe des Indexes 1" auszusprechen wünschte. Wollte ich die Theoreme dieser Arbeit nur für "Curvencomplexe, die als gegenseitiger Schnitt von $r-1$ M_{r-1}-Systemen im R_r erzeugt werden *können*," aussprechen, so könnte ich' es einem neuen Probleme anheimstellen, über die Identität aller Complexe 1. Ordnung (oder des Index 1 oder wie ich im Texte absichtlich sofort sage "linearen Complexe") mit den eben gekennzeichneten Complexen zu entscheiden.—Mit dem obigen Beweise halte ich selbstverständlich die "delicate" Frage der Rationalität aller eigentlichen Involutionen r-ter. Stufe *in linearen Räumen* für entschieden. Anders verhält es sich mit der von Castelnuovo wie von Enriques noch nicht erwähnten Frage der Involutionen $\infty^{r+1}, \ldots$ bis ∞^{2r-1} im R_r, unter denen ich die Möglichkeit irrationaler Involutionen wirklich vermuthe. Jedoch sind solche Involutionen intermediärer Stufe bisher der allgemeinen Aufmerksamkeit entgangen, cf. n. 5. am Ende dieses §1. Rational sind allgemein im R_r alle ∞^{kr}-stufigen Involutionen, ob aber auch für $k > 1$ die $\infty^{kr+k'}$-stufigen Involutionen irrational sein können, dies erst ist eine wirklich delicate Frage, die ich mit dieser Note angeregt haben will.

Denn man kann alle ∞^{rk} k-tupel des R_r auf die Punkte eines R_{rk} derart abbilden, dass jede Gesammtheit von $\infty^{r(k-1)}$ k-tupeln mit einem festen gemeinsamen Punkte P_0 auf einen $R_{r(k-1)}$ des R_{rk} abgebildet ist.

Ich bewirke dies, indem ich im R_r r lineare M_{r-1}-Systeme, jedes von k Dimensionen angebe und welche solche Lage haben, dass je r M_{r-1} aus den Systemen sich in k freien Punkten schneiden. Man kann etwa die M_{r-1}^n unicursal nehmen, etwa alle mit demselben $(n-1)$-fachen Punkte, und sieht, wie die Construction dieser r Systeme auf ein rein arithmetisches Problem hinausläuft. Dann nehme ich r lineare $R_{r(k-1)}$-Systeme, jedes mit einer Axe R_{rk-k-1} und weise sie den r M_{r-1}-Systemen zu, indem ich überdies zwischen jedem Paare eine collineare Beziehung herstelle. Dann ist die Abbildung erreicht, indem die r R_{rk-k} sich in einem Punkte schneiden. Durch einen Punkt P_0 von R_r gehen dann $r\,\infty^{k-1}$-Systeme von M_{r-1}, welchen im R_{rk} durch die r Collineationen r R_{rk-1} entsprechen werden. Diese schneiden sich in einem Raume der Dimension $r(rk-1)-(r-1)+rk=r(k-1)$, welcher $R_{r(k-1)}$ das Bild der k-tupel mit dem gemeinsamen Punkte P_0 ist.*

Durch jeden Punkt des R_{rk} gehen k solche $R_{r(k-1)}$. Die Bild-M_r einer Involution ∞^r von k-tupeln des R_r schneidet jeden $R_{r(k-1)}$ in einem Punkte und nur in diesen. Es wird in vielen Fällen möglich sein, zu bewirken, dass diese $R_{r(k-1)}$ keinen allen gemeinsamen Punkt besitzen; auch wenn die M_{r-1}-Systeme so gewählt sind, dass die $R_{r(k-1)}$ allen gemeinsame Punkte besitzen, kann der Schluss auf die Rationalität der Bild—M_r mit Sicherheit gemacht werden. Denn die M_r ist im R_{rk} ein—eindeutig auf die ∞^r-Reihe der $R_{r(k-1)}$ bezogen, ist es also auch auf die Punkte eines R_r.

Corollar. Jede Involution ∞^r im R_r ist als der gegenseitige Schnitt von je r Varietäten M_{r-1} in einem linearen ∞^r-Systeme anzusehen und zu construiren.

Ich sage: die Involution sei im M_{r-1}-Systeme erzeugt. Dies ist unendlich vielfach möglich. "Wenn eines dieser M_{r-1}-Systeme vollständig† in dem Sinne

* Für $r=2$ wurde das Theorem bei Castelnuovo Math. Ann. XLIV: "Sulla razionalità delle curve piane" bewiesen. Ich bin aber der Ansicht, dass sich der Uebertragung seines Beweises auf den R_r Schwierigkeiten entgegenstellen, die auch mit den noch unbekannten Geschlechtseigenschaften der algebraischen M_{r-1} kaum überwindbar werden. Ich hebe deswegen hervor, dass vorliegender Beweis das Schwergewicht auf ein *arithmetisches Problem* legt, die Bestimmung der restlichen Schnittpunktezahl k von M_{r-1} mit gegebener Basis und invers *die Bestimmung dieser Basis aus der Zahl k*, und dass nach Lösung dieses Problemes der übrige Beweis sich ganz natürlich entwickelt.

† Noch muss beachtet werden, dass das Theorem, welches Castelnuovo beweist (oder das von mir in Acta Math. XIX ausgesprochene) *nicht* neben den Aufsatz von Lüroth zu stellen ist. Lüroth hat sicher

ist, dass es in keinem Systeme höherer Dimension aber gleichen Ranges k enthalten ist, sind es alle." "Wenn diese Systeme unvollständig sind, ist für alle die Minimaldimension eines enthaltenden vollständigen Systemes dieselbe." Diese enthaltenden M_{r-1}-Systeme erzeugen Involutionen $\infty^{r'}$, in denen die gegebene enthalten ist.

THEOREM.—*Jeder lineare ∞^{r-1}-Complex von C_k im R_r ist der gegenseitige Schnitt von je r M_{r-1} eines linearen $\infty^{r-1} — M_{r-1}$-Systemes.*

In jedem R_{r-1} eines Büschels wird für die ausgeschnittene Involution ∞^{r-1} ein erzeugendes M_{r-2}-System (Corollar) in gemeinsamer für alle $\infty^1 R_{r-1}$ rational distincter Weise bestimmt und z. B. durch Schnitt mit $r-1$ Geraden eine M_{r-1} festgesetzt, welche die ∞^1 diese Geraden treffenden M_{r-2} enthält. Das lineare M_{r-1}-System wird durch irgend $r-1$ so bestimmte M_{r-1} combinirt.

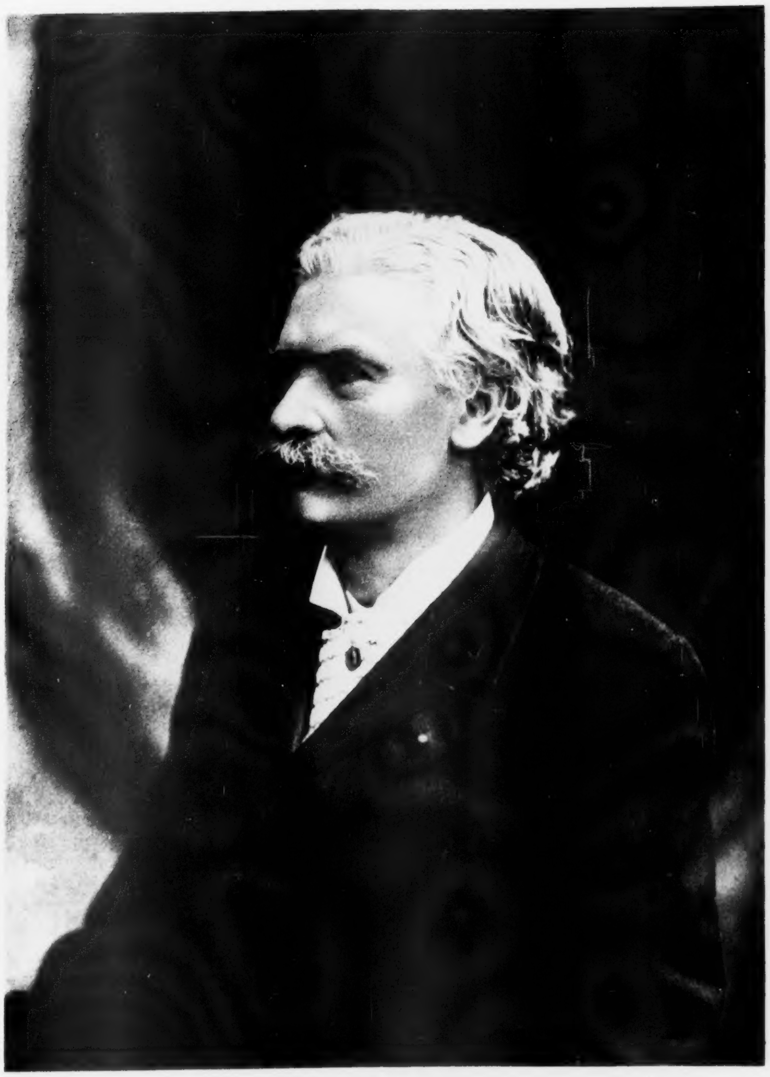
Für einen ∞^{r-1}-Curvencomplex existiren ∞ erzeugende M_{r-1}-Systeme. Ist eines "vollständig," sind es alle. Wird im Folgenden von ∞^{r-1}-Complexen von C_k gesprochen, so sind "vollständige," also solche gemeint, die nicht in $\infty^{r-1+\lambda}$-Complexen von C_k enthalten sein können.

2. Ein *eigentlicher* linearer ∞^u-Complex von C_k im R_r, wo $u > r-1$, ist ein solcher, wo eine im ganzen R_r gleiche Anzahl Punkte willkürlich angenommen werden können, sodass durch sie immer noch eine, aber nur eine Curve hindurchgehe. Für ihn muss nothwendig $u=(r-1)(d-r+2)$ sein können, ein $\infty^d - M_{r-1}$-System existiren, das ihn erzeugt. Von einem *vollständigen** Curvencomplexe kann man nur bezüglich des erzeugenden M_{r-1}-Systemes sprechen. Dies is vollständig, wenn es alle M_{r-1} mit demselben Singularitätencomplexe über der festgesetzten Basis enthält. Andererseits kommen in dieser Arbeit Fälle vor, wo ein C_k-Complex bezüglich des Geschlechtes π einer C_k als vollständig oder nicht distinguirt wird.

3. *Uneigentliche* Complexe sind der Schnitt von $r-1$ M_{r-1}-Systemen, deren Dimensionen $v_1, \ldots v_{r-1}$ von einander theilweise oder durchwegs verschieden

nicht die Absicht gehabt, *bloss* die Rationalität der Involution $a_x^n + \lambda b_x^n = 0$ zu beweisen, seine Leistung ist eine algebraisch-technische: die bekannte neuer Parameterfestsetzung. Das leistet C. nicht und habe ich im obigen Beweise überhaupt nicht zu leisten beabsichtigt.

* Das Wort "vollständig" habe ich im Hinblicke auf die Theorie der Kronecker'schen Modulsysteme gewählt, welche in neuerer Zeit die Theorie der algebraischen Mannigfaltigkeiten zu beherrschen beginnen. Für Systeme eignet es sich wohlauch besser als der Ausdruck "normal," den ich aber für einzelne Mannigfaltigkeiten im Sinne Veronese's (Math. Ann. XIX) unbedingt beibehalten möchte.



Mittag-Leffler

sind. Ist $v_1 \geqq \ldots \geqq v_{r-1}$, so muss überdies mindestens ein $v_i < (\Sigma v):(r-1) + r - 3$ sein; denn die Dimension u ist Σv und da $u = (r-1)(d-r+2)$, müssen durch $d - 1 = (\Sigma v):(r-1) + r - 3$ Punkte noch Curven gehen. Sind nun alle Curven in den M_{r-1} eines ∞^{v_r}-Systemes enthalten, so muss $v_{r-1} \geqq v_r$ sein. Im Gegenfalle ist der Complex uneigentlich. Für uneigentliche lineare Complexe gilt: *Wenn* durch eine Anzahl gegebener Punkte des Raumes nur endlich viel Curven gehen, so darf die Anzahl nicht > 1 sein.

Die—etwa $\mathfrak{S}_i$–Systeme niederster Dimension enthalten die grösste Mächtigkeit von C_k und ausser dem durch sie combinirbaren M_{r-1}-Systeme gibt es nicht M_{r-1} mit gleicher C_k-Mächtigkeit. Unter dem combinirbaren Systeme meine ich jene M_{r-1}, welche durch die $M_{r-\sigma_i}$, Schnitte der $\mathfrak{S}_i$ M_{r-1}-Systeme, hindurchgehen.

4. *Sind alle v gleich, so ist der C_k-Complex stets eigentlich.* Man nimmt zum Beweise erst zwei Systeme und indem man mit R_2 in einer Involution ∞^{2v} schneidet. Aus den zwei Systemen mögen zwei ∞^{v-1}-Systeme von Büscheln gehoben werden, sodass jede M_{r-1} in nur einem Büschel ist und beziehe sie (1, 1)-deutig auf ∞^1 Arten auf einander, sodass unter je zwei zugewiesenen Büscheln ∞^1 Projectivitäten entstehen, die ein lineares System bilden. Dann liefern je zwei projective Büschel eine M_{r-1} und diese ∞^{v+1} erzeugen das M_{r-2}-System. Mit diesem ∞^{v+1} — M_{r-1}-Systeme und dem dritten gegebenen wird ähnlich verfahren und weiter, bis alle erschöpft sind.—Man kann aber auch sofort in jedem M_{r-1}-Systeme ein $\infty^{v-r+\mu}$-System von ∞^{r-1}-S. angeben, sodass jede M_{r-1} nur in einem ∞^{r-1}-S. ist, unter ihnen (1, 1)-deutige Beziehungen, nun aber unter je zwei zugewiesenen ∞^{r-1}-S. ∞^{r-1} Collineationen so einrichten, dass jedes Paar nur in einer Collineation enthalten ist. Jede Collineation erzeugt eine M_{r-1}, deren Gesammtheit den Curvencomplex zum Schnitte hat. Die Dimension des M_{r-1}-Systemes muss ∞^{r+v-2} sein.

5. Beim Uebergange von Stralen—zu Curvencomplexen Γ muss die "Ordnung" von Γ definirt werden. Solange Γ die Dimension $u < 2r - 2$ hat, durch jeden Punkt P_1 höchstens ∞^{r-2} Curven gehen, werden ihre Tangenten in P einen Kegel bilden. Dessen Ordnung ist die Ordnung von Γ. Ist $u \geqq 2r - 2$, dann erfüllen die Curven durch P_1 i. A. den ganzen R_r, sodass durch P_2 i. A. ∞^{u-2r} Curven gehen. Ist $u \leqq 3r - 2$, so erfüllen die Tangenten in P_1, P_2 je einen Kegel. Beider Ordnungen sind gleich der Ordnung von Γ. Allgemein ist

$u > rk + k'$, wo $k' < r$, so gehen durch $P_1, \ldots. P_k$ noch ∞^{u-kr} Curven, welche in den $P_1, \ldots. P_k$ k Tangentenkegel gleicher Ordnung, der Ordnung von Γ, haben. Γ ist linear, wenn diese Kegel lineare Räume sind.

Für lineare Complexe ist zu beachten, dass nun nicht das Totalsystem aller Curven gleichen Characters einen linearen Totalcomplex bilden, wie bei den Stralen; es ist nur noch der Fall, wenn die Curve vollständiger Schnitt von $r-1$ M_{r-1} allgemeiner Natur ist. Also erst bei den Curvencomplexen ensteht die Frage nach den umfassendsten vollständigen linearen Complexen, wie es bei den Stralencomplexen der ganze Geradenraum war. *Diese umfassendsten Complexe sind der Gegenstand vorliegender Arbeit.*

Jene eigentlichen linearen Complexe, deren Dimension eine Zahl der Form $(r-1)(d-r+2)$ ist, sind der vollständige Schnitt von je $r-1$ M_{r-1} innerhalb eines linearen $\infty^d - M_{r-1}$-Systemes.

Auch die uneigentlichen Complexe müssen eine ihren characteristischen Zahlen entsprechende Dimension haben, damit sie der vollständige Schnitt von $r-1$ M_{r-1}-Systemen sein können.

Jene eigentlichen oder uneigentlichen linearen Complexe, deren Dimension nicht jene Form hat, sind in linearen höherdimensionalen enthalten, deren Dimension jene Form hat.

Jeder solche nicht vollständige lineare Complex wird erzeugt, indem man in den $r-1$ M_{r-1}-Systemen des vollständigen Verwandtschaften verschiedener Stufen constituirt und also den Schnitt der M_{r-1} auf den Schnitt solcher M_{r-1} einschränkt, welche in der Verwandtschaft zusammengehören.*

Dies gilt nicht nur für lineare, sondern auch für Complexe höherer Ordnung.—Die Ordnung von Γ hängt nicht nur von den Indices der erzeugenden M_{r-1}-Reihen, sondern auch von der Art der Verwandtschaften ab.† Ob die ihrer Dimension nach geeigneten Complexe höherer Ordnung immer als voll-

* *Diese* Erzeugung mittelst Verwandtschaften kann der vorigen Erzeugung durch vollständigen Schnitt in zwei verschiedenen Arten gegenübergestellt werden. Entweder man geht von der ersteren aus und kann dann die zweite als Degeneration auffassen, etwa wie man in der Theorie der Isomorphie die Beziehung aller Substitutionen auf alle als ausgeartete Isomorphie auffasst, oder man geht von der zweiten aus und fasst die erste als durch Einschränkung der Zuweisung aus der zweiten entstanden auf.

† Es können sogar die Curven von Γ transcendent sein; wenn nur der Complex algebraisch ist, so werden die wie im Texte construirten Kegel algebraisch sein und eine Definition der Ordnung liefern Dagegen wird man beim Uebergange zu transcendenten Complexen, wo also die gemeinten Tangentenkegel transcendent werden, nach neuen die Ordnung vertretenden Characteren suchen müssen.

ständiger Schnitt von $r-1$ M_{r-1}-Reihen erzeugt werden können, will ich hier nicht entscheiden.

Bei Curvencomplexen Γ gibt es aber ausser den Tangentenkegelordnungen noch andere characteristische Zahlen, deren Definition hier gegeben sei. Wenn $u \overline{>} 2r-2$, so wird es Curven geben, welche in P_1 einen Doppelpunkt haben, etwa ∞^{δ_2}. Dann erhält man für die Tangentenpaare dieser Curven in P einen Ort, also einen Tangentenkegel, dessen Ordnung und dessen Involution 2. Ordnung, die er trägt, für Γ auch characteristisch sind. Wächst u noch weiter, so wird es geschehen müssen, dass durch P_1 ∞^{δ_3} Curven gehen, welche P_1 dreifach haben und deren ∞^{δ_3} Tangententripel in P einen Kegel erfüllen, dessen Charactere auch für Γ characteristisch sind. Allgemein, wenn es ∞^{δ_i} Curven in Γ gibt, welche P_1 i-fach enthalten und deren Tangenten in P_1 einen Kegel erfüllen, liefert dieser wieder für Γ characteristische Zahlen.

6. *Ueber Involutionen.* Durch den Schnitt von Γ mit einen M_{r-1} entstehen folgende Begriffe: Eine Gesammtheit von ∞^{r-1} Punktgruppen in M_{r-1} heisst eine Involution, wenn jeder Punkt nur einer Gruppe angehört. Eine Gesammtheit von $\infty^{r+r'-1}$ Punktgruppen in einer M_{r-1}, wo $r'<r-2$, heisse eine *eigentliche* Involution $(r+r'-1)$. Stufe, wenn zwar nicht durch irgend r' Punkte eine Gruppe geht, wenn aber durch jeden Punkt $\infty^{r'}$ Punktgruppen bestimmt sind, welche eine $M_{r'}$ erfüllen, auf der sie eine Involution r'-ter Stufe bilden. Diese Involution heisse aber *uneigentlich*, wenn jeder Punkt von M_{r-1} durch $\infty^{r'}$ Punktgruppen ergänzt wird, die eine $M_{r''}$ erfüllen und in dieser die Involution r'-ter Stufe bilden.

Ist $r'=r-1$, so heisse die Gesammtheit der $\infty^{2(r-1)}$ Punktgruppen eine eigentliche Involution $2(r-1)$. Stufe, wenn durch irgend zwei Punkte der M_{r-1} eine Gruppe hindurchgeht und allgemein die Gesammtheit von $\infty^{f(r-1)}$ Punktgruppen ist eine eigentliche Involution $f(r-1)$-ter Stufe, wenn durch irgend f Punkte der M_{r-1} *eine* Gruppe hindurchgeht. Ist dies nicht der Fall, so heisse jene Gesammtheit eine *uneigentliche* Involution $f(r-1)$-ter Stufe. Eine Gesammtheit von $\infty^{f(r-1)+f'}$ Punktgruppen in M_{r-1} heisse eine *eigentliche* Involution der Stufe $f(r-1)+f'$, wenn durch irgend f Punkte der M_{r-1} noch $\infty^{f'}$ Gruppen bestimmt sind (= hindurchgehen), welche eine Mannigfaltigkeit von f' Dimensionen erfüllen und darin eine eigentliche Involution der Stufe f' bilden. Erfüllen aber die $\infty^{f'}$ ergänzenden Gruppen nur $f''>f'$ Dimensionen und bilden in ihr eine eigentliche oder uneigentliche Involution f'-ter Stufe so heisse die

Involution eine uneigentliche der Stufe $f(r-1)f'$. Man erkennt so, dass die Uneigentlichkeit einer Involution der Stufe $\phi = f(r-1) + f'$ durch drei Reihen von Zahlen characterisirt ist

$$\begin{array}{l} \phi, f, f_1, f_2, f_3, \dots \\ \quad f', f_1', f_2', f_3', \dots \\ \quad f'', f_1'', f_2'', f_3'', \dots \end{array}$$

wo die Gleichungen gelten:

$$\phi = f(r-1) + f', \; f' = f_1 f'' + f_1', \; f_1' = f_2 f_1'' + f_2', \; f_2' = f_3 f_2'' + f_3', \dots$$

und wo f_i'' die Dimension der Mannigfaltigkeit bedeutet, welche die Ergänzungsgruppen von $f + f_1 + f_2 + \dots + f_i$ allerdings nicht willkürlich gewählten Punkten erfüllen. Diese Punkte können nämlich überhaupt nur so gewählt werden, dass f in der M_{r-1}, f_1 in einer $M_{f''}$, f_2 in einer $M_{f_1''}, \dots, f_i$ in einer $M_{f_{i-1}''}$ variiren können, wo jedesmal die $M_{f_g''}$ in der $M_{f_{g-1}''}$ enthalten ist.

Die Zahlen f, f', f'' *sind Invarianten* nicht nur gegen birationale Transformationen sondern auch bei Uebertragung einer Involution in einem eindeutigen in eine Involution in einem mehrdeutigen Systeme.

Wird die Involution auf M_{r-1} durch den Schnitt von linearen M_{r-2}-Systemen erzeugt, ist sie also rational, so hängen die Zahlen f von den Zahlen v (§1) der M_{r-2}-Systeme ab. Es bleibt zu untersuchen, für welche Zahlenreihen f, f', f'' die Involution auf M_{r-1} nothwendig rational sein müsse und ob es im R_i und auf unicursalen M_i überhaupt nur rationale Involutionen irgend welcher Zahlen f, f', f'' gebe.

Gleichgebildete Zahlenreihen f, f', f'' gelten dann auch für jeden uneigentlichen linearen Curvencomplex höherer Stufe in R_r und allgemeiner für jeden uneigentlichen linearen M_i-Complex höherer Stufe im R_r.

Aber es gelten auch Zahlenreihen f, f', f'' gleicher Entstehungsweise für die uneigentlichen linearen Curven-oder M_i-Complexe höherer Stufe in einer M_{r-1} des R_r.

Ich unterlasse nicht, zu bemerken, dass diese Zahlen von einer so umfassenden Geltung sind, dass sie auch für Complexe auf *transcendenten* Mannigfaltigkeiten gebildet werden können.*

***Gelegentlich sei erwähnt, dass mittelst der Methoden in meiner Abh. Monatshefte 1899 auch das Problem erledigt werden kann, die uneigentlichen Involutionen höherer Stufe der Ebene zu discutiren, in denen ein System rationaler oder elliptischer Curven " enthalten " ist.**

§2.—*Lineare ∞^{r-1}-Complexe rationaler Curven.*

THEOREM I.—*Wenn die Curven ($p=0$) eines linearen ∞^{r-1}-Complexes eine einpunktig schneidente Transversalmannigfaltigkeit gestatten, so gestatten sie unendlich viele.*

Jene eine Transversale kann eine M_{r-1}, aber auch eine M_{r-2} oder $M_{r-3}, \ldots M_1$, M_0 sein. Wird dann auf jeder Curve des Systemes zu dem einzelnen Schnittpunkte mit der Transversalen eine durch den Schnitt mit einer willkürlichen M_{r-1} entstehende binäre Form hinzugenommen und irgend eine simultane lineare Covariante z. B. eine Ableitung genommen, so entsteht auf jeder C_n ein Punkt und der Ort derselben ist eine M_{r-1}, welche einpunktig schneidet.

Corollar. Ist auch nur eine der Zahlen, welche die Ordnung oder die Stützzahlen der C_n auf den Basismannigfaltigkeiten bezeichnen, ungerade, so gestattet der C_n-Complex einpunktig schneidende Transversal-M_{r-1}.

Denn in diesem Falle gibt es stets lineare Covarianten. Der Ort derselben kann sich auf eine M_i, sogar M_0 zusammenziehen, kann aber *nicht* für alle linearen Covarianten identisch verschwinden, und gibt nach Th. I. doch zu Transversal-M_{r-1} Anlass.

THEOREM II.—*Jeder lineare ∞^{r-1}-Complex rationaler Curven im R_r gestattet zweipunktig schneidende Transversal-M_{r-1} von unbegränzt hoher Dimension.*

Denn man kann zu jeder binären Form eine quadratische Covariante bilden. Schneidet man daher alle C_n durch eine M_{r-1} und bildet auf jeder C_n für die entstehende binäre Form eine bestimmte quadratische Covariante q, so ist deren Ort eine zweipunktige Transversale. Da die Ordnung der M_{r-1} beliebig hoch ist, auch simultane Formen benützt werden können, wird es immer eine Construction geben, welche nicht zu identisch verschwindenden Covarianten q führt und überdies M_{r-1} beliebig hoher Dimension liefert.

Definition.—In Uebereinstimmung mit Nöther (für M_2) kann man eine M_{r-1}, auf welcher es eine rationale Involution von ∞^{r-1} Punktepaaren gibt, eine hyperelliptische nennen. Diese Involution muss von einem linearen ∞^{r-1}-Complexe von Curven ausgeschnitten sein, da wir die Involution stets als "vollständig" voraussetzen (als Definition) und diese ist nach §1 selbst das Erzeugnis in einem linearen ∞^{r-1} — M_{r-1}-Systeme.

THEOREM III.—*Unter den zweipunktig schneidenden Transversalsystemen gibt es stets auch homaloidale oder solche, welche homaloidale enthalten.*

Die Transversal—M_{r-1}-Systeme sind sämmtlich hyperelliptisch. In der

Ebene liess sich beweisen, dass die linearen Systeme hyperelliptischer Curven stets die maximale Dimension erreichen, woraus das Enthaltensein eines homaloidalen Systemes auch gefolgert werden konnte. Ein ähnliches Theorem müsste nun auch im R_r bewiesen werden und zum selben Schlusse führen.

Aber man kann den Beweis auch darauf gründen, dass, wenn durch Vorschreibung eines Contactes höherer Ordnung in einem Punkte P aus dem Complexe der Schnittcurven ein ∞^3-Complex rationaler Curven ausgeschieden werden soll, die Verminderung der Zahl u dasselbe Gesetz befolgt wie in der Ebene. Es handelt sich also nur, die Differenz zwischen den Zahlen p und u für die Transversalensysteme zu untersuchen. Um dies bequemer thun zu können, kann man die Basisgebilde durch solche ersetzen, wo jedes nicht lineare Gebilde durch eine Anzahl linearer Gebilde ersetzt ist.

Mit Benützung dieser Zerlegung kann man übrigens den Beweis auf den des hierfolgenden Theoremes V basiren. Denn die Zahlrelationen in Frage werden durch die Zerlegung nicht geändert und da Th. V unabhängig beweisbar ist, muss das homaloidale System existiren, also die fragliche Zahlrelation bestehen.

THEOREM IV.—*Jeder lineare ∞^{r-1}-Complex rationaler Curven im R_r ist durch birationale Transformation des ganzen R_r übertragbar in einen ∞^{r-1}-Complex von Kegelschnitten.*

Benützt man eines der in Th. III erwähnten homaloidalen Systeme für eine Raumtransformation, so ensteht im R_r' der Complex von Kegelschnitten.

THEOREM V.—*Gestatten die Curven $p=0$ eines linearen ∞^{r-1}-Complexes eine 1-punktige Transversalmannigfaltigkeit, so können sie durch birationale Transformation des R_r in ein Stralbündel übertragen werden.*

Ich stelle eine Collineation unter den M_{r-1} des Systemes und den R_{r-1} durch einen Punkt O her, hiemit gleichzeitig unter den C_n und den Geraden durch O als Individuen. Dann weise ich drei 1-punktige Transversalen der C_n drei R_{r-1} im R_r' zu und setze unter jeder C_n und der zugehörigen Geraden eine Projectivität fest, in der die drei Schnittpunkte mit jenen den dreien mit diesen entsprechen. Die Gesammtheit dieser ∞^{r-1} Projectivitäten gibt die Transformation.

Corollar I. Jeder lineare ∞^{r-1}-Geraden-Complex kann durch birationale Transformation in ein Stralbündel übertragen werden. Hieraus kann geschlossen werden, dass es so viele wesentlich verschiedene Arten von linearen ∞^{r-1}-Complexen gibt, als homaloidale M_{r-1}^n-Systeme mit $(n-1)$-fachem Punkte.

Corollar II. Gestattet ein linearer ∞^{r-1}-Geraden-Complex von Curven im R_r einpunktige Transversalmannigfaltigkeiten, so haben diese solche Beschaffenheit, dass sie sich birational (im R_r) je in ein System von M_{r-1}^n mit $(n-1)$-fachem Punkte übertragen lassen.

Corollar III. Jedes Büschel einpunktiger Transversalmannigfaltigkeiten eines linearen ∞^{r-1}-Complexes kann durch birationale Transformation des R_r in ein Büschel von R_{r-1} verwandelt werden. Denn in ihnen kann man simultan auf rationale Art je ein homaloidales System von M_{r-2} festlegen und kann unter den M_{r-1} des Büschels und den R_{r-1} eines anderen Büschels in rational bestimmter Weise ∞^1 birationale Transformationen festsetzen, deren Gesammtheit die Ueberführung leistet.

Zum Schlusse sei erwähnt: Sind zwei lineare Complexe ∞^{r-1} von Curven im R_r äquivalent, so sind noch nicht irgend zwei sie erzeugende ∞^{r-1}-Systeme von M_{r-1} äquivalent. Denn ein ∞^{r-1}-Complex kann auf unendlich viele Arten durch ein lineares ∞^{r-1}-System von M_{r-1} erzeugt werden.

§3.—*Lineare ∞^{r-1}-Complexe von Kegelschnitten im R_r.*

Jedenfalls sind in einer Ebene entweder 0 oder 1 oder ∞^1 Kegelschnitte enthalten (oder in Ausnahmeebenen ∞^2).

In R_3 können die ∞^2 C_2 entweder in ∞^1 R_2 enthalten sein, welche dann ein Büschel bilden müssen oder in ∞^2. Die ∞^2 Ebenen müssen die C_2 entweder aus einem Büschel von M_2^2 oder einem Netze von M_2^2 ausschneiden. Im *ersten* Falle sind die M_2^2 des Büschels linear bezogen auf die Geraden einer rationalen Regelschaar, sodass jede M_2^2 von den sämmtlichen Ebenen durch die entsprechende Gerade in ∞^1 C_2 des Complexes geschnitten wird. Im *zweiten* Falle müssen die M_2^2 collinear auf die Ebenen bezogen sein, *damit der Complex linear werde,* weil den durch P gehenden M_2^2, welche also im Büschel sind, eine solche Mannigfaltigkeit von Ebenen entsprechen muss, von der eine einzige durch P geht. Also muss die Mannigfaltigkeit aller Ebenen ∞^2 Büschel enthalten, das heisst ein Bündel sein mit dem Scheitel O. Damit entsteht sofort das Netz von M_2^3 durch eine c_7 $p=5$. Dieses Netz ist von Montesano in den Rendiconti des Inst. Lombardo in gediegener Weise untersucht worden.

Es entstehen also: 1. C_2 in Ebenen einer Büschels, 2. C_2 in Ebenen einer rationalen Regelfläche und in M_2^2 eines Büschels, 3. C_2 in Ebenen eines Bündels und in M_2^2 eines Netzes.

Ich behaupte, dass die 2. Art birational Complexen äquivalent ist, deren C_2 eine Gerade zweimal treffen, sofern man eine biquadratische Irrationalität einführen will. Diese letztere ist nothwendig, um auf der Basisvarietät des M_2^2-Büschels einen Punkt zu individualisiren. Man bezieht dann ein Ebenenbüschel projectiv auf die M_2^2 des Büschels und projicirt von O aus jede M_2^2 auf die ihr zugewiesene Ebene, womit durch die Gesammtheit dieser ∞^1 Projectionen eine birationale Transformation im R_3 entsteht.

Uebrigens ist die 1. Art noch zu unterscheiden, je nachdem die Basispunkte der ∞^1 Kegelschnittbüschel eine irreductibele oder zwei reductibele Mannigfaltigkeiten erfüllen. Unter solcher Auffassung bleiben also nur zwei Haupttypen: I. *der Complex in einem Ebenenbüschel*, II. *der Complex in einem Ebenenbündel.**

Im R_r können die C_2, damit sie nicht auf eine einzige M_{r-1} beschränkt seien, nur entweder in ∞^{r-1} oder in ∞^{r-2} Ebenen enthalten sein. Diese Ebenen müssen jedenfalls einen Complex bilden, dessen Schubert'sche Gradzahl $[0, r] = 1$.

Jeder lineare ∞^{r-2}-R_2-Complex kann birational in die Ebenen durch eine feste Gerade verwandelt werden, wobei noch unter je zwei entsprechenden Ebenen Collineation bewirkt werden kann, sodass die C_2 wieder in C_2 verwandelt

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* Im R_3 hat Montesano Domenico in den Rendiconti dell' Accademia delle Scienze di Napoli 1895 eine Eintheilung angegeben. Dieselbe ist aber, wenn sie sich auf die birationale Transformationen des R_3 als die discriminirende Hauptgruppe bezieht, wie es auch ein Enriques'sches Citat dieser Arbeit beansprucht, *falsch*. Denn von der Involution, in welcher ein C_2-Complex die Ebenen des R_3 schneidet, hängt bezüglich der Äquivalenz gegen birationale Transformationen gar nichts ab.

Jedenfalls habe ich mich—um eines hervorzuheben—überzeugt, *dass in sehr vielen von* Montesano behandelten speciellen Congruenzen, wo die Ebenen der C_2 eine Regelfläche umhüllen, sich ein Büschel von M_2^2 herausfinden lässt, welches alle C_2 enthält, sodass Reduction auf den Typus I. des Textes möglich ist.

Ist aber ferner eine C_2-Congruenz gegeben, welche eine Ebene in einem Punktepaarsysteme eines bestimmten Typus schneidet, *so kann es unicursale Flächen geben, welche sogar homaloidal sind und von den C_2 in einem Punktepaarsysteme geschnitten werden, das sich nach räumlich birationaler Verwandlung jener Fläche in eine Ebene in ein Punktepaarsystem eines anderen Typus unsetzt, während die C_2 wieder in C_2 übergehen.*

Im Falle, wo die C_2 eine Raumcurve 4. O. $c_4\, p = 1$ vierfach treffen, ziehe man durch C_4 eine Fläche 3. O. Durch Abbildung des Schnittes derselben mit den C_2 sieht man, dass er von derselben Natur ist, wie jener durch die Schnitte der Stralen eines Büschels O mit den C_3 eines Büschels durch O entstehende. Dieses Punktepaarsystem ist intricat unterschieden von dem, welches in einer Ebene von den C_2 über C_4 ausgeschnitten wird. Es gibt ferner eine birationale Transformation, welche die F_3 in eine Ebene und die C_2 wieder in C_2 verwandelt und daher zwei Congruenzen einander *äquivalent* macht, welche die *Ebenen* des R_3 in zwei *verschiedenen Typen* von Involutionen durchschneiden. Und so in anderen Fällen!

werden.* Die Basispunkte der ∞^{r-2} C_2-Büschel können dann noch entweder eine irreductibele Mannigfaltigkeit M_{r-2}^n, oder zwei Mannigfaltigkeiten $M_{r-2}^{n'}$, $M_{r-2}^{n''}$ erfüllen, von denen die erste die Scheitelgerade der Ebenen zur $(n-4)$-fachen, die anderen sie zur $(n'-2)$, $(n''-2)$-fachen Geraden haben.

Beim ∞^{r-1}-Complexe von Ebenen ist nun zu unterscheiden, aus welchem M_{r-1}^2-Systeme niederster Dimension der Ebencomplex die C_2 ausschneidet. Es kann dies ein ∞^1 bis ∞^{r-1}-System sein. Zum ∞^i-Systeme gehört dann für jede M_{r-1}^2 ein lineares ∞^{r-i-1}-System von R_2, welche also je eine M_{r-i+1} erfüllen. Diese ∞^i M_{r-i+1} müssen ein lineares System bilden und ihre Gesammtheit muss mittelst ihrer R_2 einen linearen ∞^{r-1}-R_2-Complex geben. Für $i = r-1$ entsteht jedenfalls auch im R_r ein ∞^{r-1}-System von M_{r-1}, welche durch den Schnitt von je $r-1$ die C_2 hervorbringen.

Constructionen von C_2-Complexen ∞^{r-1} kann man mannigfach ersinnen. Einen ∞^{r-2}-Complex von C_2 im R_{r-1} projicire man aus einem Punkte O auf einen R'_{r-1} des R_r, lasse dann O in einer rationalen Curve variiren und den R'_{r-1} in einem Büschel im R_r. Dann entsteht ein ∞^{r-1}-Complex des R_r. Allgemeiner man construire in den sämmtlichen R_i durch einen R_{i-1} des R_r je einen linearen ∞^{i-1}-Complex von Kegelschnitten einer der drei vorerwähnten Typen; die Gesammtheit gibt einen ∞^{r-1}-C_2-Complex des R_r. Der extremste Fall dieser Reihe ist dann der Fall $i-2$, unser obiger erster Typus. Man kann diese Complexe der besonderen Beschaffenheit der Focal-M_{r-2} wegen als singulär bezeichnen.

Eine andere beachtenswerthe Classe ist der Complex der C_2, die eine M_{r-2}^2 des R_r in zwei Punkten schneiden.† Er entsteht durch Projection aus einem Punkte O einer M_r^2 des R_{r+1} von allen C_2 eines in M_r^2 durch die R_2 irgend eines linearen ∞^{r-1}-Complexes entstandenen Schnittcomplexes.

§4.—*Lineare $\infty^{u>r-1}$-Complexe rationaler Curven im R_3.*

Lemma I. Jedes lineare M_{r-1}-System mit $u = r$ und rationalen gegenseitigen Schnittcurven ist ein homaloidales System.

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* Jede C_2 muss die Focal-M_{r-2} in $2r$ Punkten schneiden (Darboux, "Leçons sur les surfaces II"). Da vier auf die Basispunkte des Büschels in der Ebene von C_2 entfallen, so folgt, dass die den R_2 gemeinsame Gerade $(r-2)$-fach für die M_{r-2} ist.

† Mario Pieri, Rendiconti Ist. R. Lombardo hat diesen Complex im R_4 in vortrefflicher Weise behandelt.

Im R_3 entsteht durch den gegenseitigen Schnitt in jeder M_2 des Systemes ein vollständiges ∞^2-System rationaler Curven in jeder M_2 des Systemes, also, indem die bekanntlich daraus folgende Abbildbarkeit benützt wird, ein homaloidales System, das ist vom Range 1. Es ist also auch das M_2-System vom Range 1, liefert also eine birationale Transformation durch die Beziehung auf die R_2 eines R_3'.

Indem man so weiter schliesst, beachte man zunächst im R_r, dass aus der Existenz eines homaloidalen Systemes auf einer M_{r-1} deren Abbildbarkeit auf einen R_{r-1} sofort folgt, weil es einen linearen ∞^{r-1}-Complex von Curven zu finden sofort möglich ist, die jene M_{r-1} in je einem Punkte schneiden, der Complex aber direct den Punkten eines R_{r-1} zugewiesen ist.

Ist dann aber auf M_{r-1} ein vollständiges lineares ∞^{r-1}-System von M_{r-2} gegeben, die sich gegenseitig in rationalen Curven schneiden, so kann der Rang nicht > 1 sein, weil schon für den Rang 2 die Dimension $> r-1$ folgen würde, oder die Abhängigkeit der Punktepaare auf den Schnittcurven, was für $p=0$ derselben nicht möglich ist. Also folgt auch aus der Existenz eines *vollständigen linearen ∞^{r-1}-Systemes von M_{r-2}* auf M_{r-1} die Abbildbarkeit.

In dem im Lemma vorausgesetzten Systeme entsteht nun wegen $u=r$ auf jeder M_{r-1} des Systemes ein vollständiges lineares ∞^{r-1}-System von M_{r-2}, woraus die Abbildbarkeit wegen der rationalen Schnittcurven und der Rang 1 für das ganze M_{r-1}-System folgt.

THEOREM VI.—*Für jedes lineare ∞^u-System, $u > r-1$,* von M_{r-1} mit rationalen gegenseitigen Schnittcurven gibt es entweder ∞^1 einpunktige Transversalen oder ∞^2 zweipunktige Transversalen der Curven.*

Auf jeder M_2 des Systemes entsteht im R_3 ein ∞^{u-1}-System rationaler Curven. Die M_2 sind für $u > r$ rational distinct auf die Ebene abbildbar und indem man die Theoreme über die ∞^{u-1}-Systeme in der Ebene verwendet,† wo $u-1>1$ zu nehmen ist, ist ersichtlich, dass es entweder ein rational distinctes System von ∞^1 rationalen Curven gibt, welche für alle ∞^{u-1} Curven 1-punktige Transversalen sind oder ein ∞^2-System rationaler Curven, welche zweipunktige Transversalen sind. Dieses ∞^2-System ist ebenfalls rational distinct und überdies homaloidal. Dazu kommt ein dritter Fall, wo das ∞^{u-1}-Sys-

* Wir bedürften eigentlich nur mehr der Theoreme für $u > r$, da das Lemma den Fall eines vollständigen $u=r$ erschöpft.

† Monatshefte für Math. und Phys. 1899.

tem dem System von ∞^3 C_2 durch ein Punktepaar äquivalent ist, wo das System selbst als 2-punktiges Transversalensystem für sich aufgefasst werden kann. In jedem dieser Fälle ist also das Transversalensystem auch auf der M_2 rational distinct.

Als solches muss es stets auf allen M_2 des Systemes gleichzeitig durch ein lineares M_2-System ausgeschnitten werden können. Dies ist in den beiden vorherigen zu verwendenden Fällen um so sicherer, als das Transversalensystem in der Ebene, nämlich für die Typen das Geradenbüschel oder das Geradennetz in dem Curvensysteme selbst als Bestandtheil enthalten ist, also auch auf den M_2 des Systemes entsprechend Bestandtheil sein muss.

Es gibt also ein lineares M_2-System, welches alle Curven des Complexes entweder in einem oder zwei Punkten schneidet. Dieses System ist überdies in dem gegebenen als Bestandtheil enthalten. Von hier ab schliessen wir nun ganz ebenso weiter für die M_3 im R_4, bis wir zum R_r gelangen.

Corollar. Wenn es keine einpunktigen Transversalflächen im R_3 gibt, kann die Dimension des M_2-Systemes nicht >6 sein, und sie ist entweder genau gleich 6 oder 4.

Denn für das auf einer M_2 des Systemes erzeugte Curvensystem gilt von der Ebene her, dass die Dimension bei zweipunktigen Transversalen entweder gleich 5 oder 3 ist.

THEOREM VII.—*Jedes lineare ∞^u-System von M_{r-1} mit erzeugtem Complexe rationaler Curven und einem Büschel einpunktiger Transversal-M_{r-1} für dieselben ist birational transformirbar in ein System, dessen erzeugter Curvencomplex Curven C_n sind, die einen R_{r-2} in $n-1$ Punkten schneiden.*

Ich nehme durch Vorschreibung einfacher Punkte $P_1, \ldots P_{u-r}$ aus dem Systeme ein ∞^r-System heraus. Dasselbe wird nach dem Lemma ein homaloidales System sein. Ich transformire durch dieses System in einen R_r'. Dann erscheinen statt der $\infty^{2(r-1)}$ Curven $\infty^{2(r-1)}$ Geraden. Die einpunktigen Transversalen sind jetzt in M_{r-1} verwandelt, welche alle Geraden des R_r' in je einem Punkte treffen, sind also R_{r-1} eines Büschels. Diese $\infty^1 R_{r-1}$ müssen auch alle transformirten C_n in je einem variabeln Punkte treffen; ihre Axe R_{r-2} muss also $n-1$ Punkte mit jeder C_n gemeinsam haben.

Das transformirte M_{r-1}-System muss von jedem dieser R_{r-1} in einem homaloidalen Systeme geschnitten werden, sodass, obzwar $u>r$ sein mag, dennoch in jedem der ∞^1 Transversal-R_{r-1} kein höher dimensionales als ein ∞^{r-1}-

System erscheinen darf, da kein M_{r-2}-System mit $u > r - 1$ den Rang 1 haben kann. Dann lässt sich leicht unter den $\infty^1 R_{r-1}$ des R_r und $\infty^1 R'_{r-1}$ eines Büschels in R'_r eine Collineation und unter je zwei entsprechenden R'_{r-1}, R_{r-1} eine birationale Transformation herstellen, welche jenes Schnittsystem in die $\infty^{r-1} R_{r-2}$ verwandelt. Somit sind die M_{r-1} in solche verwandelt, welche einen $(n-1)$-fachen R_{r-2} besitzen.

Auch wenn wir gar nicht die Dimension 1 für die einpunktigen Transversalen als aus der Ebene bekannt* voraussetzen wollen, können wir dieselbe erschliessen. Denn wird nur die Voraussetzung gemacht, das überhaupt 1-punktige Transversalen vorhanden sind, so führt die Benützung eines im ∞^u-Systeme enthaltenen homaloidalen ∞^r-Systemes nothwendig zu einem M_{r-1}-Systeme in R'_r, wo die Transversalen in R'_{r-1} verwandelt sind. Wären dieselben nun ∞^{r-1} oder ∞^{r-2} .... ∞^2, so wären die Curven des Complexes in C_n verwandelt, welche $(n-1)$-fach einen R_0 oder R_1, ... R_{r-3} treffen würden, daher entweder jede in einem R_2 oder in einem R_3, ... R_{r-1} enthalten wäre. Normalcurven sind aber die einzigen, welche in dieser Weise einen "eigentlichen" Complex bilden können, wie eine leichte Ueberlegung aus der Definition des "eigentlichen" Complexes folgert.

THEOREM VIII.—*Jedes lineare ∞^u-System, $u > r-1$, von M_{r-1} mit rationalen gegenseitigen Schnittcurven, welche mehr als ∞^1 Transversal-M_{r-1} einpunktigen Schnittes zulassen, kann räumlich birational transformirt werden in ein System von M_{r-1}, die sich in Normalcurven* 2., 3., 4., ... $(n-1)$. *Ordnung schneiden, welche einen festen R_0, R_1, R_2,* .... oder *R_{n-3} in resp.* 1, 2, 3, ... $(n-2)$ *Punkten treffen.*

Solche Curvencomplexe können aber einfach construirt werden. Von den gemeinsamen Secanten-R_i aus projicire man sie auf einen R_{r-1}, R_{r-2}, R_{r-3}, .. R_2 resp. in einen Geradencomplex. Daher:

THEOREM IX.—*Soll ein eigentlicher ∞^u-Complex von rationalen Curven einem der Typen des Theoremes IV äquivalent sein, so kann seine Dimension nicht* $> 2(r-2)$ *sein.*

In diesem Falle wird sich über jeder Geraden des R_{r-i} (Projectionsraumes) nur eine Normalcurve befinden. In jedem R_{i+1} durch den Scheitel-R_{i-1} *kann dann noch nach* Willkür die Normalcurve M_1^{i+1} construirt werden, sofern dieselbe nur einem für alle R_{i+1} einheitlichen Gesetze unterworfen ist. [Wird in jedem R_{i+1} ein $\infty^{v>0}$-System von Curven construirt, so wird der Complex uneigentlich.]

* Monatshefte für Math. und Phys. 1899.

THEOREM X.—*Gestattet ein eigentlicher Complex rationaler Curven nur zweipunktige Transversalen, so hat er entweder als erzeugendes M_{r-1}-System ein solches der Dimension $r + 1$ oder der Dimension $r + 3$.*

Denn im Corollare zu Theorem I. wurde bereits geschlossen, dass im R_3 die Dimension 6 oder 4 ist. Nun ist im M_{r-1}-System des R_r jedes erzeugte M_i-System wieder von derselben Art, was die Transversalen anlangt. Daher werden die M_3-Systeme in den M_4 entweder die Dimension 7 oder 5 haben, u. s. w. bis zu den M_{r-1}.

THEOREM XI.—*Gestattet ein eigentlicher Complex rationaler Curven nur zweipunktige Transversalen, so ist er birational transformirbar entweder in das System der Kegelschnitte, in denen sich die M^2_{r-1} durch eine feste M^2_{r-2} schneiden, oder in das System der Curven* 4. O., *Schnitte von M^2_{r-1}, welche sich längs eines R_{r-3} berühren mit demselben Tangenten-R_{r-1} in allen Punkten.*

Denn im Falle der Dimension $r + 3$ besteht ein ∞^2-System von Transversal-M_{r-1}, welche sich wechselseitig in M_{r-2} so treffen müssen, dass diese M_{r-2} die M_2 des gegebenen M_{r-1}-Systemes in je einem Punkte schneiden. Wird nun aus dem gegebenen ∞^{r+3}-Systeme wieder durch einfache Punkte ein homaloidales System herausgenommen und wird mit diesen in R'_r transformirt, so entsteht aus dem Transversalensystem jetzt ein M_{r-2}-System im R'_r, das die Geraden des R'_r in Punktepaaren schneiden muss, also aus M^2_{r-1} besteht. Die M^2_{r-1} haben überdies die Eigenschaft, sich zu zweien in abbildbaren M_{r-2} zu treffen. Bei dieser Transformation gehen die Curven des gegebenen Complexes in solche über, welche von den M^2_{r-1} in zwei variabeln Punkten getroffen werden. Ich werde beweisen, dass *dieses* M^2_{r-1}-Netz stets birational in ein R_{r-1}-Bündel verwandelt werden könne. Ich nehme aus dem ∞^{r+3}-M_{r-1}-System ein ∞^{r-1}-System heraus und beziehe dasselbe collinear auf ein ∞^{r-1}-System von R'_{r-1} im R'_r, gleichzeitig das ∞^2-System von Transversalen auf ein ∞^2-System von R'_{r-1} im R'_r reciprok.

Dann kann, da die M^2_{r-2} die M_2 in je einem Punkte schneiden, hierdurch eine birationale Transformation unter R_r, R'_r bestimmt werden, welche die $\infty^{r-1}M_{r-1}$ und gleichzeitig die ∞^2 Transversalen in R_{r-1} verwandelt. Die Curven des Complexes sind in solche verwandelt, welche von ∞^{r-2} R_{r-1} durch eine Gerade in 4 und von ∞^2 R_{r-1} durch einen Punkt O in 2 variabeln Punkten geschnitten werden. Die erzeugenden M^n_{r-1} haben in O einen $(n-1)$-fachen Punkt. Statt hieran die Discussion zu knüpfen, verwende ich eine dritte Art von Transformation.

Ich beginne mit R_3. Aus dem Vorhergehenden ist der Nutzen zu ziehen, dass wir ∞^2 einpunktige Transversalcurven der sämmtlichen $\infty^k M_2$ kennen. In Folge dessen können wir aus dem ∞^6-System ein ∞^3-System $(M_2^n$ auswählen, welches die M_2 in ∞^3 Curven eines linearen Systemes vom Range 2 schneidet, während das ganze ∞^5-System vom Range 4 ist. Die gegenseitigen Schnittcurven dieser $(M_2^n$ sind zweipunktige Transversalen. Ich nehme nun abermals das Hilfs-∞^3-System aus dem ∞^k-Systeme, welches eine birationale Tr. und daher im R_3 aus dem vorher bezeichneten ∞^3-Systeme ein solches mit C_2 als erzeugten Curven liefert und hebe das hervor, dass in Folge dessen die Schnittcurven unter den $(M_2^n$ rational sind. Da es aber ein vollständiges ∞^3-System ist, so ist es nach dem Lemma ein homaloidales System. Benütze ich nun dieses für eine birationale Transformation, so verwandle ich endlich die M_2^n des gegebenen Systemes in M_2^2 und die Curven des Complexes in C_4. Jetzt folgt von selbst, dass die M_2^2, damit die C_4 rational werden, sich in einem Punkte O berühren müssen. In der That bilden diese M_2^2 ein ∞^6-System. Wir entdecken nun auch das bei dem ersten Beweisansatze gefundene ∞^2-System 2-punktiger Transversalen; das sind die Ebenen durch O.

Bemerken wir jetzt, dass für dieses System in R_3 stets ein homaloidales System vorhanden ist, das die C_n in 4 Punkten trifft und rational distinct ist; es sind die R_2 des R_3, Bestandtheile der M_2^2. Aber auch irgend drei M_2^2 schneiden sich in 4 Punkten.

Im R_4 haben wir nun auf den M_3 des Problemsystemes das ∞^6-System. Wir haben wieder für die M_3 einpunktige Transversalen, eben in Folge des rational distinct vorhandenen homaloidalen ∞^3-Systemes, das als Bestandtheil im ∞^6-Systeme enthalten und daher zweifellos durch ein ∞^3-System von M_3 ausschneidbar ist, dessen gegenseitigen Schnittcurven die einpunktigen Transversalen sind. Mit Hilfe dieser Transversalencurven können wir aus dem ∞^6-Systeme von M_2 auf den M_3 ein ∞^4-System von $(M_2^n$ auswählen, welches sich nur mehr in Curven schneidet, die zweipunktige Transversalen des gegebenen Systemes sind. Durch Benützung eines beliebigen, im gegebenen Systeme enthaltenen homaloidalen Systemes, welches im R_3 jene Schnittcurven in C_2 verwandelt, weisen wir die Rationalität derselben, deswegen aber die Homaloidität von $(M_2^2$ nach, transformiren mittelst dieses Systemes und erhalten wieder M_2^2, welche einen Complex von Curven C_4 bestimmen. Die sämmtlichen Ränge dieses Systemes müssen 4 sein. Daraus schliesse ich, dass sich die M_2^2 längs einer Geraden

berühren und auch in allen Punkten derselben einen und derselben Tangenten-R_3 haben.

In derselben Weise kann bis zum R_r geschlossen werden, dass die sammtlichen Ränge 4 sind, um endlich, da hiezu weder gemeinsame M^4_{r-3} noch zweimal gezählte gemeinsame M^2_{r-3} dienen, auf ein System von M^2_{r-1} zu gelangen, welche sich längs aller Punkte eines R_{r-3} berühren. Sie sind also von $r = 6$ Kegel mit einem Doppel-R_{r-6}, der innerhalb R_{r-3} variirt.

Es bleibt noch der Fall der Dimension $r+1$. Dann ist das gegebene M^n_2-System vom Range 2 in Betreff der Punkte, aber durch successive Schlüsse ersieht man, dass auch alle übrigen Ränge 2 sind. Durch Vorschreibung eines einzigen Punktes P wird also ein ∞^r-System vom Range 1, das ist ein homaloidales System, ausgeschieden. Wird durch dieses in einen R'_r transformirt, so wird nothwendig ein ∞^{r+1}-System von M^2_{r-1} entstehen, dessen sämmtliche Ränge 2 sind. Das einzige derartige System ist das im Theoreme genannte.

Bevor ich den Beweis verlasse, betone ich nochmals, dass der Angelpunkt desselben das im § anfangs gebrachte Lemma ist. Ich fasse alle Resultate zusammen in das folgende:

THEOREM XII.—*Jeder eigentliche lineare Complex rationaler Curven im R_r kann räumlich birational in einen der folgenden Typen übertragen werden:*

I. *Das Stralenbündel in einem Scheitel O.*

II. *Einen ∞^{r-1}-Complex von Kegelschnitten.*

III. *Einen $\infty^{2(r-1)}$-Complex von rationalen Normcurven i. O. mit einem festen $(i-1)$-punktigen Sehnen-R_{i-2}, $i \leqq r-1$ oder einen in diesem einthaltenen unvollständigen Complex.*

IV. *Einen Complex von C_n mit einem gemeinsamen $(n-1)$-punktigen Sehnen-R_{r-1}, wobei n willkürlich gegeben sein kann, Schnitt innerhalb eines linearen Systemes M^m_{r-1} mit $(m-1)$-fachem R_{r-2}.*

V. *Den Complex der Kegelschnitte, welche eine feste M^2_{r-2} zweipunktig treffen, oder einen in diesem enthaltenen unvollständigen Complex.**

VI. *Den Complex der Curven 4. Ordnung, in denen sich die M^2_{r-1} schneiden, welche einen festen R_{r-3} enthalten und längs desselben gemeinsame Tangenten-R_{r-1} besitzen, oder einen in diesem enthaltenen unvollständigen Complex.*

Um den zweiten Passus unter IV. zu beweisen, bemerke ich, dass ein R_{r-1} durch R_{r-2} die M^m_{r-1} in zwei M_{r-2} schneiden muss, welche nur einen gemeinsamen Punkt besitzen. Ist also das System nicht homaloidal im R_{r-1}, was nur

*** Durch diesen Zusatz wird die Gesammtheit der Geraden im *Rr* erledigt.**

bei einem ∞^{r-1}-Complexe (I. oder II.) eintreten könnte, so müssen die M_{r-2} Ebenen, also der M_{r-2} muss $(m-1)$-fach sein.

Um in VI. zu erkennen, dass die sämmtlichen Ränge 4 sind, hat man zu beachten, dass z. B. zwei M_3^2 im R_4, die sich längs einer Geraden g berühren, eine Schnitt-M_2^4 haben durch g^2, deren beide Tangentenebenen in einem Punkte P von g in dem gemeinsamen Tangenten-R_3 der M_3^2 in P enthalten sind.

Aus diesem Grunde zählt g im Schnitte von M_2^4 mit einer weiteren M_3^2 des Systemes 4-statt nur zweifach, und ähnlich im R_r. Die 2^{i-1} Tangenten-R_i in einem Punkte P von R_{r-3} sind jedesmal in dem Tangenten-R_{r-1} an alle M_{r-1}^2 in P enthalten.

Die Typen des Aequivalenztheoremes VIII. beweisen folgendes für unser Transversalenprincip wichtige Theorem:

THEOREM XIII.—*Für jeden eigentlichen linearen* ∞^u-*Complex rationaler Curven im* R_r *gibt es entweder:*

1. *ein rational distinctes Büschel einpunktig schneidender* M_{r-1} (*welche abbildbar sind*) *und gleichzeitig einen* ∞^{2r-3}-*Complex rationaler die erzeugenden* M_{r-1} *des Complexes einpunktig schneidender Curven, oder*

2. *ein rational distinctes lineares* ∞^i-*System einpunktig schneidender* M_{r-2}, *aber nur im Falle, wo* $u \leqq 2(r-1)$ *ist, oder*

3. *ein rational distinctes und überdies homaloidales und im erzeugenden Systeme enthaltenes* ∞-*System von* M_{r-1}, *welche die* C_n *zweipunktig schneiden, wo dann stets* $u = 2r$ *ist, oder*

3. *ein rational distinctes, weil im erzeugenden Systeme als Degeneration enthaltenes* ∞^r-*System von* M_{r-1}, *welche die* C_n *zweipunktig schneiden.*

In 3. und 4. gibt es einen rational distincten ∞^{r-1}-Complex rationaler Curven, welche die erzeugenden M_{r-1} sämmtlich einpunktig schneiden.

Jedes einzelne dieser Systeme von Transversalen ist daraus zu erschliessen, dass bei birationaler Uebertragung Dimension, Geschlecht und Anzahl der Transversalpunkte in einem Systeme invariant sind. Es ist von Wichtigkeit, zu bemerken, dass die Transversalen 1.–4. auch für einen enthaltenen unvollständigen Complex rational distinct bleiben.

§5.—*Verallgemeinerung des Picard'schen Satzes auf* R_r.

Das Theorem XIII. leitet zu folgendem weiteren Theoreme, dessen speciellsten Fall Herr Picard in Cr. J. 100 gegeben hat (für $r=3$):

THEOREM XIV.—*Die* M_{r-1} *des* R_r, *welche rationale ebene Schnittcurven besitzen, sind:*

I. *die* M_{r-1}^2,

II. *die* M_{r-1}, *welche aus einer rationalen Reihe von* R_{r-2} *zusammengesetzt sind,*

III. *die* M_{r-1}^4, *welche Kegel sind, die aus einem* R_{r-3} *eine Steinersche Fläche* 4. *O. projiciren.*

Aus der Existenz der $\infty^{3(r-2)}$ ebenen Curven folgt zunächst die Abbildbarkeit. Der algebraische Beweis hat keine Schwierigkeit. Geometrisch stützt man sich am besten auf das *Lemma:* Enthält eine M_{r-1} einen ∞^{r-2}-Complex von rationalen Curven, der den Index 1 besitzt, und der an sich eine unicursale Mannigfaltigkeit ist, so ist M_{r-1} unicursal.

Um diesen Satz zu beweisen, transformire ich mittelst einer einfachen geometrischen Construction die M_{r-1} in eine solche, wo die ∞^{r-2} Curven in einem ∞^{r-1}-Complexe rationaler Curven des R_2 enthalten sind, die den ganzen Raum einfach erfüllen. Dann wende ich das Princip an, dass die verschiedenen Geschlechtszahlen sich nicht ändern, wenn die Basisgebilde des Complexes so zerfallen, dass er in ein Stralbündel transformirbar wird. Ist er dies aber, so wird M_{r-1} ersichtlich eine Kegelfläche, die von einem R_0 eine unicursale M_{r-2} projicirt.

Nun sind die einzigen M_{r-2}-Systeme, welche sich in Curven $p=0$ schneiden und als Bilder einer M_{r-1} im R_r dienen können, die in IV. V. VI. des Th. XII genannten.

§6.—*Die uneigentlichen linearen Complexe rationaler Curven im* R_3.

THEOREM XV.—*Im* R_3* *ist jeder uneigentliche Complex von rationalen Curven* C_n *durch birationale Transformation des* R_3 *in einen Complex von ebenen Curven in den Ebenen eines Büschels (wenigstens irrational) übertragbar, sobald es ein alle* C_n *enthaltendes* M_2*-Büschel gibt.*

Das zweite erzeugende M_2-System hat mindestens $u=2$. Ist u genau $=2$, so sind die Systeme von C_n in den M_2 des Büschels homaloidale Netze, die Schnittcurven der M_2 des Netzes unter einander einpunktige Transversalcurven

* Während die eigentlichen Complexe rationaler Curven im R_3, für welche $u>2$, von F. Enriques in Math. Ann. Bd. 46 auf drei Typen zurückgeführt sind, welche aus 4. 5. 6. des Theoremes XIII für $r=3$ hervorgehen, hat er auch im R_3 die uneigentlichen Complexe gar nicht erwähnt. Uebrigens ist seine im R_3 benützte Methode nicht nur grundverschieden von der meinigen, sondern ihr in gewissem Sinne gerade entgegengesetzt.

jener M_2, wobei jedoch von einfachen Basispunkten des zweiten Systemes abgesehen wird. In der That ändert die Weglassung derselben nichts an der Rationalität oder Ordnung der Schnittcurven und das System kann mit Bezug auf diese beiden Charactere erst nach Weglassung der einfachen Basispunkte als vollständig angesehen werden.* Ist $u > 2$, so kann aus dem ∞^u-Systeme von C_n durch Vorschreibung von Punkten, welche in der Basis des M_2-Büschels enthalten sind, ein homaloidales Netz, das ist also aus dem zweiten M_2-Systeme ein vollständiges ∞^2-System ausgeschieden werden, dessen gegenseitige Schnittcurven einpunktige Transversalen der M_2 des Büschels sind. Wird nun dieses Netz auf ein Ebenenbündel reciprok und das erste M_2-Büschel auf ein Ebenenbüschel projectiv bezogen, so hat man eine birationale Transformation nach einem R_3' vermittelt, welche das M_2-Büschel in ein Ebenenbüschel überführt.

Es wird aber sogar gleichzeitig das transformirende Netz so gewählt werden können, dass es entweder das einpunktig schneidende Büschel enthält oder selbst das zweipunktig schneidende Netz ist und daher entstehen durch die Transformation dann die beiden Typen von Curvensystemen in den einzelnen Ebenen.

Soll rein rational transformirt werden, dann dürfen die einfachen Basispunkte nicht verwendet werden. Aber es ist zu ersehen,† dass auf jeder M_3 des Büschels entweder ein ∞^1-System 1-punktiger Transversalen also insgesammt ein ∞^2-System 1-punktiger Transversalen des anderen M_2-Systemes erscheint, was Transformirbarkeit dieses in ein monoidales System beweisen würde oder ein ∞^2-oder ∞^3-System 2-punktiger Transversalen, was im einen Falle rein rationale Transformirbarkeit auf ein Ebenenbüschel oder im anderen Falle Transformirbarkeit *auf ein Büschel von M_2^2 bewirkt.* Aber eine weitere kleine Discussion beweist, dass diese nie vollständig in einem Büschel sein können.

THEOREM XVI.—*Die Complexe des Theoremes XV. sind äquivalent:* 1. *Curven C_n, welche einen festen $(n-1)$-fachen Punkt haben und in Ebenen eines Büschels enthalten sind oder* 2. *Curven C_2, welche in den R_2 eines Büschels sind und eine C_n, welche dessen Axe $(n-2)$-punktig trifft, zweifach treffen oder* 3. *alle C_2, welche eine feste Gerade zweipunktig treffen.*

Es bleibt nur zu n. 1. zu erklären, dass man die bei der Wahl der Transformation in den einzelnen Ebenen entstandenen $(n-1)$-fachen Punkte durch

*Siehe §4, Lemma I.

†Denn die Flächen mit einem rational distincten linearen ∞^2-oder ∞^3-System rationaler Curven sind rein rational abbildbar auf eine Ebene oder eine M_2^2.

neuerliche Anwendung einer Collineation auf je eine der Ebenen, das ist also durch eine neuerliche räumlich birationale Transformation in einen und denselben Punkt O der Büschelaxe verlegen kann.

THEOREM XVII.—*Wenn im R_3 die niederste Dimension eines enthaltenden M_1-Systemes $v_2 > 1$ ist, so kann der C_n-Complex in den Schnitt aller M_2^2 mit allen Ebenen oder in den Complex aller Schnittcurven der Monoide $M_2^m(O^{m-1})$ mit den Kegelflächen eines linearen Systemes in O verwandelt werden, welche selbst rational sind.*

Es ist dann eine Dimension v_1 (cf. §1) mindestens 3. Jedes der beiden Systeme besteht aus abbildbaren Flächen, da jede dieser Flächen ∞^2 rationale Curven eines linearen Systemes enthält. Aus dem Systeme ∞^{v_2} wähle ich ein ∞^2-System durch Vorschreibung einfacher Punkte, welche auch in der Basis des zweiten Systemes enthalten sind. Dann schneidet es diese M_2 in homaloidalen Netzen und daher sind die gegenseitigen Schnittcurven jenes ∞^2-Systemes rationale Curven, welche die M_2 des zweiten zu einpunktigen Transversalen haben. Verwandelt man sie in ein Stralenbündel, so kann dies stets auch so geschehen, dass die ∞^2 M_2 in ∞^2 Ebenen verwandelt werden. Das zweite System geht in ein M_2-System über, dessen M_2 von ∞^2 Ebenen eines Bündels O in rationalen Curven geschnitten werden und die Geraden durch O zu einpunktigen Transversalen haben, also in $M_2^n(O^{n-1})$.

Die Curven müssen auch von O aus durch ein lineares System von Kegelflächen projicirt werden, weil sie auf jeder einzelnen $M_2^n(O^{n-1})$ des Systemes ein lineares System bilden, und diesem in der Abbildung auf R_2, welche durch die Projection aus O entsteht, ein lineares System entsprechen muss.

Wenn die sämmtlichen Flächen gleichzeitig in Ebenen verwandelt werden, und die Kegelflächen, welche wir eben erwähnt haben, sind Kegelflächen 2. O., so benützen wir gleich das ganze System der Kegelschnitte, welche Schnitte der M_2^2 mit allen R_2 sind.

Es wurde implicite die Annahme gemacht, dass Basispunkte des zweiten Systemes vorhanden sind. Diese Annahme gilt nur dann nicht, wenn diese M_2 sämmtlich schon die R_2 oder die M_2^2 sind.

Gerade diese Bemerkung führt zu einer zweiten Art von Transposition. Wir mögen $v_2 > 2$ voraussetzen. Solange das zweite System Basispunkte (in hin reichender Anzahl) hat, können wir nun, da die einfachen Punkte P nichts an der Rationalität der Schnittcurven des ersten Systemes ändern können, mindes-

tens ein ∞^3-System im ersten enthalten voraussetzen, dessen gegenseitige Schnitte rationale Curven sind, welches also, wenn es durch einfache Punkte als ein relativ vollständiges erhalten wurde oder es an und für sich vollständig ist, nothwendig homaloidal sein muss. Die Transformation durch dieses System liefert für das zweite M_2-System ein solches, dessen sämmtliche ebenen Schnitte rational sind. Nach einem Theoreme (Picard's) für den R_3 können diese nur sein: 1. die sämmtlichen M_2^2 oder ein darunter enthaltenes System, 2. Regelflächen, 3. Steinerische Flächen. Für 1. ist zu erwähnen, dass die M_2^2 nur von den R_2 in einem Complexe von Curven mit $p = 0$ geschnitten werden.

Für n. 2. sage ich nun, dass die Regelflächen die vielfachen Curven gemeinsam haben müssen, also dass ihre Erzeugenden entweder eine feste Gerade schneiden müssen, die für alle $(n - 1)$-fach ist, oder eine Congruenz der Ordnung 1 bilden, welche also birational in ein Stralbündel übertragbar ist, somit die Regelflächen in Kegelflächen mit gemeinsamen Scheitel verwandelt werden. Die Steiner'schen Flächen sub 3. können, obzwar es lineare Systeme von Monoiden gibt* keine linearen System bilden, ohne dass die Doppelgeraden fest sind, was den vorigen Fall gibt. So bleiben die Typen des Theoremes XVI. Dieselben gehörig interpretirt geben nun:

THEOREM XVIII.—*Für jeden uneigentlichen linearen Complex rationaler Curven im R_3 gibt es entweder:*

1. *ein rational distinctes ∞^1-System von M_2, welche alle Curven einpunktig treffen, oder*

2. *ein rational distinctes ∞^2-System von M_2, welche alle Curven des Systemes zweipunktig treffen, oder*

3. *ein rational distinctes und überdies homaloidales ∞^3-System von M_2, welche alle Curven zweipunktig treffen.*

Im Falle n. 1. gibt es einen ∞^2-Complex der Ordnung 1 von rationalen Curven, welche die beiden erzeugenden Systeme einpunktig treffen (eventuell dem einen ganz angehören); im Falle n. 2. gibt es ausser den gegenseitigen Schnittcurven jenes M_2-Systemes keine Curven, welche die M_2 auch nur in zwei Punkten treffen würden. Das eine erzeugende M_2-System ist ein Partialsystem (Degeneration) des anderen.

Diese Schlüsse beruhen alle darauf, dass durch birationale Transformation des R_3 ein im Sinne des Geschlechtes oder Punktranges vollständiges System wieder in ein vollständiges System übergeht.

* Mario Pieri, Giorn. di Matematich'e vol. XXXI, beweist wohl zuerst die Existenz von "Büscheln" von Monoiden. Es gibt aber auch höhere lineare Systeme von Monoiden im R_3 und dann auch im R.

§7.—*Die uneigentlichen linearen Complexe rationaler Curven im R_r.*

Lemma II. Wenn im R_r ein vollständiges System von M_{r-1} eine M'_{r-1} in einem vollständigen Systeme schneiden soll, so darf es keine einfachen Basispunkte ausserhalb M'_{r-1} besitzen. Denn die Weglassung dieser würde nichts an der Rationalität ändern, ebensowenig an den $r-2$ ersten Rängen des M_{r-1}-Systemes, würde aber die Dimension vergrössern.

Lemma III. Die einzigen linearen Systeme im R_r, welche keine Basispunkte besitzen und von allen Ebenen in rationalen Curven geschnitten werden, sind die M^2_{r-1}. Die beiden anderen Arten M_{r-1} aus Theorem XIII. können ohne Basispunkte kein lineares System erschöpfen.

THEOREM XIX.—*Wenn im R_r alle erzeugenden Systeme $\Sigma_1, \ldots \Sigma_{r-1}$ Dimensionen $> r-1$ haben, also (Bezeichnung §1) $v_{r-1} > r-1$,* so kann der Complex rationaler Curven räumlich birational verwandelt werden entweder* 1. *in einen Complex von Kegelschnitten, Schnitt eines linearen Systemes von M^2_{r-1} mit einem linearen Systeme von R_{r-1}, oder* 2. *in den Schnitt eines linearen Systemes von Monoiden $M^n_{r-1}(O^{n-1})$ mit einem linearen Complexe von Kegelflächen zweier Dimensionen mit dem Scheitel in O.*†

Σ_{r-1} ist entweder nach Lemma aus §4 homaloidal oder es kann ein homaloidales System herausgenommen werden. R_r hiedurch in R'_r transformirt liefert Systeme $\Sigma_1, \ldots \Sigma_{r-2}$ von Picard'schen Mannigfaltigkeiten des R'_r. Die 3. Art kann keine linearen Systeme liefern, weil es keine linearen Systeme von C_4 mit Doppelpunkten gibt, deren drei Doppelpunkte nicht fest wären. Schneidet man die linearen Systeme von M_{r-1} der 2. Art mit Räumen R_3, so erhält man lineare Systeme von Regelflächen und nach dem in §6 bei Theorem XVI Gesagten folgt:

Lineare Systeme von M_{r-1} mit ∞^1 R_{r-2} können nur mit festem R_{r-2}, der von allen erzeugenden R_{r-2} in R_{r-3} geschnitten wird oder in einem ∞^2-Systeme der Ordnung 1 von R_{r-2} existiren. Das letztere kann räumlich birational in das System der R_{r-2} durch einen festen R_{r-3} transformirt werden.

* Die Dimensionen v_i des Systemes Σ_i werden sich stets auf als vollständig vorausgesetzte Systeme beziehen.

† Es treten hier zum ersten Male die linearen Complexe von M_2 auf. Es ist klar, dass sich auch für solche Complexe müssen Aequivalenztheoreme herleiten lassen, entsprechend den von Nöther, Bertini und mir (Monatshefte) für Curven gefundenen. Aber es scheint, dass diese Aequivalenztheoreme in ihrem Kerne doch nichts weiter als die Aequivalenztheoreme für die Curven enthalten werden. Dies ist in Uebereinstimmung mit dem, was ich in der Einleitung über die Grundlagen der Theorie der Transformationsgruppen gesagt habe und *was sich sogar auf stetige Gruppen überträgt.*

Wird eines der Σ_i so transformirt, so können die anderen, welche die rationalen C_n ausschneiden, nur so beschaffen sein, dass sie Monoide sind und den festen R_{r-2} oder R_{r-3} zum Scheitel haben.

THEOREM XX.—*Wenn $v_{r-1}=1$ und in den M_{r-1} von Σ_{r-1} eigentliche Complexe von Curven entstehen, so kann der Complex räumlich birational verwandelt werden entweder* 1. *in einen Complex von Kegelschnitten, die eine feste M_{r-3}^2 treffen oder* 2. *in C_4, welche von den M_{r-2}^2 durch einen festen R_{r-4} mit gegenseitiger Berührung erzeugt werden, oder* 3. *in C_n, welche einen festen $R_{r-2}(n-1)$-punktig treffen,* 4. *in Normcurven C_i, welche einem $R_i(i-1)$-punktig treffen und in allen diesen Fällen sind die Curven des typischen Complexes nur in den ∞^1 R_{r-1} eines Büschels enthalten.*

Das Theorem ist die Verallgemeinerung von XIX. und kann Schritt für Schritt in gleicher Weise bewiesen werden. Es ist also das Schwergewicht auf den Nachweis der einpunktigen Transversalcurven aller M_{r-1} von Σ_{r-1} zu werfen und es sind nachträglich die einzelnen R_{r-1} des Büschels collinear so zu transformiren, dass die festen Sehnenräume aus den einzelnen R_{r-1} in den Schnitt-R_{r-2} übergehen und dort coincidiren.

Aehnliche Theoreme liessen sich für $v_{r-1}=2, \ldots, r-1$ geben und es wird nützlich sein, sie abzuleiten; ich wende mich aber Raummangels wegen zum allgemeinen Theoreme, das etwa so ausgesprochen werden kann.

THEOREM XXI.—*In jedem uneigentlichen Complexe rationaler Curven des R_r gibt es, wenn $r>3$, entweder* 1. *ein rational distinctes und überdies homaloidales ∞^r-System von M_{r-1}, welche für alle Curven des Complexes zweipunktige Transversalen sind oder* 2. *ein rational distinctes und überdies homaloidales ∞^r-System von M_{r-1}, welche für alle Curven vierpunktige Transversalen sind oder* 3. *ein rational distinctes ∞^1 oder $\infty^2 \ldots \infty^{r-1}$-System von M_{r-1}, welche für alle Curven einpunktige Transversalen sind. Im letzteren Falle gibt es stets einen linearen ∞^{r-1}-Complex von Curven, welche gleichzeitig für sämmtliche erzeugenden Systeme einpunktige Transversalen sind.*

Der Beweis soll nun die Giltigkeit für R_{r-1} voraussetzen, also auch für jeden in einer abbildbaren M_{r-1} enthaltenen Complex und von da aus den R_r einbeziehen, verkettet sich aber in eigenthümlicher Weise mit dem Beweise des Aequivalenztheoremes.

Sei $v_{r-1} \leqq r-1$; jede M_{r-1} von Σ_{r-1} wird einen eigentlichen oder uneigentlichen Complex des C_n enthalten. Für jeden gilt das Theorem XXI. per hypothesin, also ist es möglich, aus den übrigen Σ_i Untersysteme auszuscheiden,

welche sich in einem ∞^{r-1}-Systeme einpunktiger Transversalen der M_{r-1} schneiden.

Es mögen nun die $M_{r-v_{r-1}+1}$, in denen sich die M_{r-1} des Σ_{r-1} gegenseitig schneiden, collinear auf die $R_{r-v_{r-1}+1}$ bezogen werden, welche durch einen festen $R_{r-v_{r-1}}$ eines R_r gehen. Dann kann mittelst der einpunktigen Transversalen von vorhin jede der $M_{r-v_{r-1}+1}$ auf den entsprechenden $R_{r-v_{r-1}+1}$ birational bezogen werden und zwar, da auch in jeder $M_{r-v_{r-1}+1}$ das Theorem XXI gilt, so, dass die Transversalen dieses Theoremes in $R_{r-v_{r-1}}$ verwandelt werden. Alle diese einzelnen auf rational distincte Art festlegbaren Transformationen geben eine birationale Transformation des R_r, durch welche also bereits die M_{r-1} von Σ_{r-1} in R_{r-1} verwandelt sind und überdies in jedem einzelnen—weil in jedem einzelnen der letzten Schnitt $R_{r-v_{r-1}+1}$—gewisse typische Curvencomplexe hergestelt sind. Dann bildan aber die R_{r-1} des R'_r das im Theoreme XXI. behauptete Transversalensystem der Numern 1. und 2. und es existirt im Falle 3. nothwendig ein rational distinctes Transversalensystem M_{r-1}, welches die M_{r-1} in den rational distincten Transversalsystemen M_{r-1} schneidet.

Für $v_{r-1} > r-1$ folgt die Giltigkeit von XXI. im R_r aus Theorem XIX. Der Beweis des Th. XXI. hat nun gleichzeitig das folgende Aequivalenztheorem geliefert.

THEOREM XXII.—*Jeder uneigentliche lineare ∞^u-Complex rationaler Curven C_n im R_r ist räumlich birational äquivalent entweder:*

I. *einem Complexe von Kegelschitten, welche in einem linearen Systeme von R_{r-1} enthalten sind, das auch durch das lineare System ihrer Schnitt-R_i vertreten sein kann,**

II. *einem Complexe von Kegelschnitten, welche in einem linearen Systeme von M^2_{r-1} enthalten sind, oder*

III. *einem Complexe von C_4, welche in einem linearen Systeme von R_i mit*

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* Es gibt verschiedene Constructionen von Kegelschnittcomplexen, die zwar recht speciell sind, aber ihrerseits Anknüpfungspunkte für neue Fäden geometrischer Untersuchung bieten.

Im R_5 verwende man drei windschiefe Geraden g_1, g_2, g_3, die Ebenen, die ihnen incident sind, bilden einen linearen ∞^3-Complex. Diesen beziehe man collinear oder rational auf ein ∞^3-System von M^2_{r-1}-Systemen, jedes der Dimension (oder Mächtigkeit) ∞ , $u \geq 1$, so erhält man durch Schnitt jeder Ebene des Complexes mit dem entsprechenden $\infty^u M^2_{r-1}$-Systeme ein lineares ∞^u-System von M^2_1 in der Ebene. Wenn insbesondere $u = 1$, ensteht ein ∞^4-Complex. Der Typus wird gefunden, indem man räumlich birational den ∞^3-R_2-Complex in alle Ebenen durch einen R_1 verwandelt. Die Herstellung von Kegelschnitten in diesen Ebenen geschieht durch das Transversalenprincip.

festem R_{i-1} *enthalten sind und ihre Doppelpunkte in einem festen* R_{i-3} *des letzteren besitzen, oder*

IV. *einem Complexe von Normalcurven* C_i, *welche in einem linearen Systeme von* R_{r-1} *enthalten sind, und einem* R_i, *der allen gemeinsam ist, zum* $(i-1)$*-punktigen Sehnenraume haben, oder*

V. *einem Complexe von Curven, welche der Schnitt eines linearen Systemes von Monoiden* $M^m_{r-1}(O^{m-1})$ *mit einem Complexe rationaler zweidimensionaler Kegelflächen sind.*

Wird auf diesen Complex V. eine Transformation angewendet,* welche den Schnitt der Kegelflächen mit einem allgemeinen R_{r-1} wieder auf die typische Form bringt, dabei aber die Geraden durch O, resp., wenn O ein linearer Raum R_ω ist, $R_{\omega+1}$ durch R_ω wieder in solche verwandelt, so entsteht das folgende absichtlich von XXII. getrennt gehaltene Theorem:

THEOREM XXIII.—*Jeder uneigentliche lineare Complex rationaler* C_n *im* R_r, *welcher einpunktige Transversal-*M_{r-1} *besitzt, kann räumlich birational verwandelt werden in den Schnitt von* $r-1$ M_{r-1}*-Systemen* Σ_i *der Ordnungen* $m_1, \ldots. m_{r-1}$, *welche bezüglich* (m_1-1)*-fach,*† (m_2-1)*-fach,* .... $(m_{r-1}-1)$*-fach einen Raum* $O_{i_1}, \ldots. O_{i_{r-1}}$ *enthalten, wo* $i_1 \geqq i_2 \geqq \ldots. \geqq i_{r-1}$ *und überdies jeder* O_{i_λ} *in dem* $O_{i_{\lambda-1}}$ *enthalten ist, resp. mit ihm coincidirt, wenn* $i_\lambda = i_{\lambda-1}$ *und wo eines der Systeme* Σ_{i-1} *nothwendig ein System von* $(r-1)$*-dimensionalen Kegeln ist.*

Schlussbemerkung. Neben dem allerdings tiefer gehenden Aequivalenzprobleme, das Gegenstand dieser Arbeit ist, gibt es doch auch noch andere Untersuchungsrichtungen auf diesem Gebiete; z. B: Man soll alle projectiv verschiedenen linearen Complexe (die eigentlichen und uneigentlichen) von Raumcurven 3. O. im R_3 bestimmen—oder von Raumcurven 4. O. 2. Sp. im R_3—oder von Normcurven r. O. im R_r—oder von Normcurven i. O. $p=0$ im R_r—oder von rationalen Raumcurven $(r+1)$. O. im R_r etc.

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* Es wird nämlich das Lemma: Gibt es eine birationale Transformation T im R_{r-1} so gibt es im R_r eine birationale Transformation T_1, welche die Stralen eines Bündels mit Scheitel O gemäss T unter einander verwandelt, einfach durch Anwendung von T auf die R_{r-1} eines Büschels bewiesen. Wenn es nun eine birationale Transformation im R_{r-1} gibt, welche einen Complex ∞^u von rationalen Curven in einen anderen überträgt, und dies auch dann, wenn der zweite Curvencomplex der typische ist, dessen Kenntnis im R_{r-1} bei unserem Beweise vorausgesetzt werde. Diese Transformation ist es nun, von welcher an dieser Stelle des Textes Anwendung gemacht wird.

† Darunter ist also auch der Fall enthalten, wo ein lineare Complex von rationalen zweidimensionalen Regelflächen mit den sämmtlichen R_{r-1} zum Schnitte gebracht wird.

Transformation of Systems of Linear Differential Equations.

By E. J. Wilczynski.

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§1. Staeckel has shown* that the most general transformation, which converts a general homogeneous linear differential equation of order $m > 1$, into another of the same form and order, is

$$T: \quad x = f(\xi), \quad y = \phi(\xi)\eta,$$

where $f(\xi)$ and $\phi(\xi)$ are arbitrary functions of ξ. If $m = 1$ the most general transformation is

$$x = f(\xi), \quad y = \phi(\xi)\eta^{\lambda},$$

where λ is a constant.

In this paper we consider, more generally, a *system* of linear differential equations, and find the most general transformation which converts such a system into a system of the same order. The transformation thus found, of course, contains T as a special case. The method of investigation is essentially the same as that of Staeckel.

A theory of invariants of systems of linear differential equations, based on this general transformation, is now being worked out by the writer.

§2. Any system of n independent homogeneous linear differential equations, containing n unknown functions $y_1, y_2, \dots y_n$ of x, and their derivatives of

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* Crelle's Journal für Mathematik, Bd. 111.

order 1, 2, ... m, can be written in the form

$$\left.\begin{array}{l} y_i^{(m)} + \sum\limits_{k=1}^{n} (p_{m-1,\,i,\,k}\, y_k^{(m-1)} + \cdots + p_{1ik}\, y_k' + p_{0ik}\, y_k) = 0, \\ \qquad\qquad (i = 1,\ 2.\ \ldots\ \lambda_1), \\ y_j^{(m-1)} + \sum\limits_{k=1}^{n} (p_{m-2,\,j,\,k}\, y_k^{(m-2)} + \cdots + p_{1jk}\, y_k' + p_{0jk}\, y_k) = 0, \\ \qquad\qquad j = (\lambda_1 + 1,\ \lambda_1 + 2,\ \ldots\ \lambda_1 + \lambda_2), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_\sigma' + \sum\limits_{k=1}^{n} p_{0\sigma k}\, y_k = 0, \quad (\sigma = \lambda_{m-} + 1,\ \lambda_{m-1} + 2,\ \ldots\ \lambda_m), \\ \qquad \sum\limits_{k=1}^{n} p_{0\tau k}\, y_k = 0, \quad (\tau = \lambda_m + 1,\ \lambda_m + 2,\ \ldots\ \lambda), \\ \qquad\qquad \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_m + \lambda = n. \end{array}\right\} \tag{1}$$

The integration of such a system involves

$$\lambda_1 m + \lambda_2 (m - 1) + \cdots + \lambda_m$$

arbitrary constants, and as such we can take the values of

$$\left.\begin{array}{l} y_1 \ \cdots\ y_{\lambda_1},\ y_1',\ \cdots\ y_{\lambda_1}',\ \cdots\ y_1^{(m-1)},\ \cdots\ y_{\lambda_1}^{(m-1)}, \\ y_{\lambda_1+1},\ \cdots\ y_{\lambda_1+\lambda_2}\ \cdots\cdots\cdots,\ y_{\lambda_1+1}^{(m-2)},\ \cdots\ y_{\lambda_1+\lambda_2}^{(m-2)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{\lambda_1+\lambda_2+\cdots+\lambda_{m-1}+1},\ \cdots\cdots\cdots\ y_{\lambda_1+\lambda_2+\cdots+\lambda_{m-1}+\lambda_m} \end{array}\right\} \tag{I}$$

for an arbitrary value x_0 of x.

§3. We wish to find the most general transformation

$$y_i = g_i(\xi;\ \eta_1,\ \eta_2,\ \ldots\ \eta_n), \quad x = f(\xi;\ \eta_1,\ \eta_2,\ \ldots\ \eta_n), \tag{2}$$

which will transform (1) into a system of the same form (1′), which we can imagine written down, if in (1) we substitute Greek letters η, π and ξ for y, p and x.

Now from (2) we obtain

$$\left.\begin{aligned}
\frac{dy_i}{dx} &= \frac{\dfrac{\partial g_i}{\partial \xi} + \sum\limits_{\lambda=1}^{n} \dfrac{\partial g_i}{\partial \eta_\lambda}\, \dfrac{d\eta_\lambda}{d\xi}}{\dfrac{\partial f}{\partial \xi} + \sum\limits_{\lambda=1}^{n} \dfrac{\partial f}{\partial \eta_\lambda}\, \dfrac{d\eta_\lambda}{d\xi}} = \frac{Y_{i1}}{\sigma}, \\
\frac{d^2 y_i}{dx^2} &= \frac{\sigma \dfrac{dY_{i1}}{d\xi} - Y_{i1} \dfrac{d\sigma}{d\xi}}{\sigma^3} = \frac{Y_{i2}}{\sigma^3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d^\mu y_i}{dx^\mu} &= \frac{Y_{i\mu}}{\sigma^{2\mu-1}},
\end{aligned}\right\} \tag{3}$$

where
$$\sigma = \frac{\partial f}{\partial \xi} + \sum_{\lambda=1}^{n} \frac{\partial f}{\partial \eta_\lambda}\, \frac{d\eta_\lambda}{d\xi}, \tag{4}$$

and Y_{i1}, Y_{i2}, etc., are defined by these equations, and are rational integral functions of $\eta'_\lambda, \eta''_\lambda, \ldots\, \eta^{(\mu)}_\lambda$. In particular,

$$Y_{i2} = \sigma \frac{dY_{i1}}{d\xi} - Y_{i1} \frac{d\sigma}{d\xi},$$

and if we denote by $H_{i2\lambda}$ the coefficient of η''_λ in Y_{i2},

$$H_{i2\lambda} = \frac{\partial g_i}{\partial \eta_\lambda}\, \frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta_\lambda}\, \frac{\partial g_i}{\partial \xi} + \sum_{\mu=1}^{n} \left(\frac{\partial g_i}{\partial \eta_\lambda}\, \frac{\partial f}{\partial \eta_\mu} - \frac{\partial f}{\partial \eta_\lambda}\, \frac{\partial g_i}{\partial \eta_\mu} \right) \eta'_\mu . \tag{5}$$
$$(\lambda = 1, 2, \ldots\, n).$$

These cannot all vanish identically, for else the functions f and g_i would not be dependent.

If we differentiate
$$\frac{d^{\mu-1} y}{dx^{\mu-1}} = \frac{Y_{i,\mu-1}}{\sigma^{2\mu-3}},$$
with respect to x, we get
$$\frac{d^\mu y}{dx^\mu} = \frac{\sigma \dfrac{dY_{i,\mu-1}}{d\xi} - (2\mu - 3) \dfrac{d\sigma}{d\xi} Y_{i,\mu-1}}{\sigma^{2\mu-1}},$$

so that
$$Y_{i\mu} = \sigma \frac{dY_{i,\mu-}}{d\xi} - (2\mu - 3) \frac{d\sigma}{d\xi} Y_{i,\mu-1}. \tag{6}$$

Denoting generally by $H_{i\mu\lambda}$ the coefficient of $\eta_\lambda^{(\mu)}$ in $Y_{i\mu}$, we have, therefore,

$$H_{i\mu\lambda} = \sigma H_{i,\,\mu-1,\,\lambda},$$

and hence

$$H_{i\mu\lambda} = \sigma^{\mu-2} H_{i2\lambda}, \qquad (\mu = 2, 3, \ldots . m), \tag{7}$$

so that $H_{i\mu\lambda}$ is different from zero, if $H_{i2\lambda}$ is.

§4. If we substitute the values (2) and (3) in (1), we get

$$\left.\begin{aligned} Y_{im} + \sum_{k=1}^{n} (p_{m-1,\,i,\,k} Y_{k,\,m-1}\sigma^2 + p_{m-2,\,i,\,k} Y_{k,\,m-2}\sigma^4 + \ldots . \\ + p_{1ik} Y_{k1}\sigma^{2m-2} + p_{0ik}\, g_k\, \sigma^{2m-1}) = 0, \\ \text{etc.} \ldots . \text{etc.} \qquad\qquad\qquad\qquad \end{aligned}\right\} \tag{8}$$

Now Y_{im} is linear in $\eta_1^{(m)}, \ldots . \eta_n^{(m)}$, and actually contains at least one of the m^{th} derivatives, since at least one of the coefficients $H_{i2\lambda}$ and, therefore, at least one $H_{im\lambda}$ is different from zero.

Now, it must be possible to solve (8) for λ_1 derivatives of order m, λ_2 of order $m - 1$, etc. For the transformed system is to be of the same order as (1). If the notation is so chosen, we shall, therefore, be able to express $\eta_1^{(m)}, \ldots . \eta_{\lambda_1}^{(m)}$ in terms of lower derivatives, etc. Moreover, these expressions must be linear and homogeneous in $\eta_1, \ldots . \eta_n$ and their derivatives, so that the transformed system may be of the same form as (1). If, then, in (8) itself, we divide each equation by the coefficient of one of the derivatives of highest order which occurs in it, the resulting equations must themselves be homogeneous and linear in $\eta_1, \ldots . \eta_n$, etc.

Let $\eta_\lambda^{(m)}$ be a derivative which occurs in the i^{th} equation (8). Divide by $H_{im\lambda}$. Since the coefficients p_{abc} are arbitrary, each of the terms

$$\frac{Y_{k,\,m-1}\, p_{m-1,\,i,\,k}\, \sigma^2}{H_{m\lambda}}, \ldots . \frac{Y_{k1}\, p_{1ik}\, \sigma^{2m-2}}{H_{im\lambda}}, \quad \frac{p_{0ik}\, g_k\, \sigma^{2m-1}}{H_{im\lambda}} \tag{9}$$

must be homogeneous and linear in $\eta_1, \ldots . \eta_n, \eta_1', \ldots . \eta_n'$, etc.

Now the last of these expressions is

$$\frac{p_{0ik}\,[f(\xi;\, \eta_1, \ldots . \eta_n)]\; g_k\,[(\xi;\, \eta_1, \ldots . \eta_n)] \left(\dfrac{\partial f}{\partial \xi} + \displaystyle\sum_{\lambda=1}^{n} \dfrac{\partial f}{\partial \eta_\lambda}\dfrac{d\eta_\lambda}{d\xi}\right)^{m+1}}{\dfrac{\partial g_i}{\partial \eta_\lambda}\dfrac{\partial f}{\partial \xi} - \dfrac{\partial f}{\partial \eta_\lambda}\dfrac{\partial g_i}{\partial \xi} + \displaystyle\sum_{\mu=1}^{n}\left(\dfrac{\partial g_i}{\partial \eta_\lambda}\dfrac{\partial f}{\partial \eta_\mu} - \dfrac{\partial g_i}{\partial \eta_\mu}\dfrac{\partial f}{\partial \eta_\lambda}\right)\dfrac{d\eta_\mu}{d\xi}}.$$

This is linear and homogeneous in $\frac{d\eta_\lambda}{d\xi}$, when $m > 1$, only if

$$\frac{\partial f}{\partial \eta_\lambda} = 0, \qquad (\lambda = 1, 2, \ldots . n),$$

i. e., if x is a function of ξ only, say

$$x = f(\xi).$$

In that case, the expression

$$\frac{p_{0ik}[f(\xi)]\, g_k(\xi; \eta_1, \ldots . \eta_n) f'(\xi)^m}{\frac{\partial g_i}{\partial \eta_\lambda}}$$

must be a homogeneous linear function of $\eta_1, \ldots . \eta_n$. Moreover, the notation can always be so chosen that $\lambda = i$, so that $\eta_i^{(m)}$ occurs in the i^{th} equation (8). Then

$$\frac{p_{0ik}[f(\xi)]\, g_k(\xi; \eta_1, \ldots . \eta_n) f'(\xi)^m}{\frac{\partial g_i}{\partial \eta_i}} \quad \begin{pmatrix} i = 1, 2, \ldots . \lambda_1 \\ k = 1, 2, \ldots . n \end{pmatrix} \tag{10}$$

must be linear and homogeneous in $\eta_1, \ldots . \eta_n$.

A similar examination of the $m - 2^{\text{nd}}$ expression (9),

$$\frac{Y_{k2}^{\cdot}\, p_{2ik}\, \sigma^{2m-4}}{H_{imi}},$$

shows that this expression is linear in $\eta_1', \ldots . \eta_n'$ if, and only if,

$$\frac{\partial^2 g_k}{\partial \eta_\lambda \partial \eta_\mu} = 0, \qquad (\lambda, \mu = 1, 2, \ldots . n),$$

so that

$$g_k = \alpha_{k1}(\xi)\eta_1 + \alpha_{k2}(\xi)\eta_2 + \ldots . + \alpha_{kn}(\xi)\eta_n + \alpha_{k0}(\xi).$$

But since (10) must be linear and homogeneous in $\eta_1, \ldots . \eta_n$, α_{k0} must vanish.

In the general case $m > 1$, we have, therefore, shown that the most general transformation which converts (1) into a system of the same order and degree is

$$\left. \begin{array}{c} x = f(\xi), \quad y_k = \alpha_{k1}(\xi)\eta_1 + \alpha_{k2}(\xi)\eta_2 + \ldots . + \alpha_{kn}(\xi)\eta_n, \\ (k = 1, 2, \ldots . n), \end{array} \right\} \tag{11}$$

and moreover, it is clear that every transformation of this form actually effects the required change, provided that

$$|\alpha_{ki}(\xi)| \neq 0.$$

§5. We still have to examine the particular case $m = 1$. But in this case it is better to adopt an entirely different method of proof.

Let

$$\frac{dy_k}{dx} = p_{k1} y_1 + \ldots\ldots + p_{kn} y_n \qquad (k = 1, 2, \ldots\ldots n) \quad (12)$$

be the given system. Let

$$y_{k1}, y_{k2}, \ldots\ldots y_{kn} \qquad (k = 1, 2, \ldots\ldots n) \quad (13)$$

be a simultaneous fundamental system of (12), so that the general solutions will be

$$y_k = c_1 y_{k1} + c_2 y_{k2} + \ldots\ldots + c_n y_{kn}, \qquad (k = 1, 2, \ldots\ldots n).$$

Any fundamental system of (12) is then of the form

$$\overline{y}_{ki} = c_{i1} y_{k1} + c_{i2} y_{k2} + \ldots\ldots + c_{in} y_{kn}, \quad (i, k = 1, 2, \ldots\ldots n), \quad (14)$$

where

$$\Delta = |c_{ij}| \neq 0, \qquad (i, j = 1, 2, \ldots\ldots n),$$

and c_{ij} are arbitrary constants.

But we can write (12) in a different way. If we differentiate each equation (12) $n - 1$ times, we get

$$\frac{d^\lambda y_k}{dx^\lambda} = p_{k\lambda 1} y_1 + \ldots\ldots + p_{k\lambda n} y_n, \quad (\lambda = 1, 2, \ldots\ldots n). \quad (15)$$

By eliminating the $n - 1$ quantities y_i, $i \neq k$ from (15), we obtain n equations

$$r_{kn} \frac{d^n y_k}{dx^k} + r_{k, n-1} \frac{d^{n-1} y_k}{dx^{n-1}} + \ldots\ldots + r_{k0} y_k = 0 \quad (16)$$
$$(k = 1, 2, \ldots\ldots n)$$

for $y_1, \ldots\ldots y_n$. In special cases these equations may be of lower than the n^{th} order, but in general they are not.

But there are relations between the equations (16), so that the general solution of one of them, say y_1, being known, the others are at once obtained in the form

$$y_k = s_{k0} y_1 + s_{k1} \frac{dy_1}{dx} + \ldots\ldots + s_{k, n-1} \frac{d^{n-1} y_1}{dx^{n-1}}, \quad (17)$$
$$(k = 2, 3, \ldots\ldots n).$$

This follows simply from (15) by putting $k=1$ and solving for $y_2, \ldots . y_n$. In other words, *equations* (16) *are cogredient*, as might also have been proved from the fact that the simultaneous fundamental systems of (12) are cogredient.

The general system (12) is, therefore, equivalent to the system

$$\left.\begin{aligned} & r_n \frac{d^n y_1}{dx_n} + r_{n-1}\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots . + r_0 y_1 = 0, \\ & y_k = s_{k0} y_1 + s_{k1}\frac{dy_1}{dx} \quad + \ldots . + s_{k,\,n-1}\frac{d^{n-1}y_1}{dx^{n-1}}, \quad (k=2,\,3,\ldots . n). \end{aligned}\right\} \quad (18)$$

If the transformation

$$y_k = g_k(\xi\,;\,\eta_1, \ldots . \eta_n), \quad x = f(\xi\,;\,\eta_1, \ldots . \eta_n) \qquad (19)$$

converts (12) into a system of the same form and order

$$\frac{d\eta_k}{d\xi} = \pi_{k1}\eta_1 + \ldots . + \pi_{kn}\eta_n, \quad (k=1,\,2,\ldots . n), \qquad (12a)$$

it must also convert (18) into a system of the same form and order

$$\left.\begin{aligned} & \rho_n \frac{d^n \eta_1}{d\xi^n} + \rho_{n-1}\frac{d^{n-1}\eta_1}{d\xi^{n-1}} + \ldots . + \rho_0 \eta_1 = 0, \\ & \eta_k = \sigma_{k0}\eta_1 + \sigma_{k1}\frac{d\eta_1}{d\xi} \quad + \ldots . + \sigma_{k,\,n-1}\frac{d^{n-1}\eta_1}{d\xi^{n-1}}, \end{aligned}\right\} \qquad (18a)$$

(18a) being equivalent to (12a) just as (18) is to (12).

But, by the method of the general case, it is now seen that the only transformations which convert (18) again into a linear system are, in general,

$$y_k = \alpha_{k1}(\xi)\,\eta_1 + \ldots . + \alpha_{kn}(\xi)\,\eta_n, \quad x = f(\xi),$$

provided that $n > 1$, and every such transformation does change (12) into a system of the same form.

For $n = m = 1$ the theorem is not true. Staeckel has settled this case. For particular cases, of course, there may be other transformations which effect the required transformation. For instance, if $m=1$, $n>1$, $p_{ik}=0$ for $i \neq k$ and $p_{kk} \neq 0$, transformations of the form

$$x = f(\xi), \quad y_k = \phi_k(\xi)\,\eta_k^{\lambda_k},$$

where λ_k is a constant, are also permissible.

§6. We have the general theorem. *All transformations which convert a general system of n homogeneous linear differential equations into another of the same form and order, have the form*

$$x = f(\xi), \quad y_k = \alpha_{k1}(\xi)\eta_1 + \dots + \alpha_{kn}(\xi)\eta_n, \quad (k = 1, 2, \dots n),$$

where f, $\alpha_{k1} \dots \alpha_{kn}$ *are arbitrary functions of* ξ. *Only if* $n = 1$, *and if the single differential equation to which the system then reduces is of the first order, is there an exception, the most general transformation being in that case*

$$x = f(\xi), \quad y = \alpha(\xi)\eta^{\lambda},$$

where λ *is a constant.*

This theorem can be extended to systems of non-linear homogeneous differential equations by a method analogous to the method employed for this purpose by Staeckel in the case of a single differential equation.

UNIVERSITY OF CALIFORNIA, BERKELEY, *March* 8, 1900.

Distribution of the Ternary Linear Homogeneous Substitutions in a Galois Field into Complete Sets of Conjugate Substitutions.

By L. E. Dickson.

The substitutions of the general linear homogeneous group G_m on m indices with coefficients in the $GF[p^n]$ may be classified into complete sets of conjugate substitutions by applying the general theorems given in an earlier paper (*American Journal*, vol. XXII, pp. 121–137). The classification is based upon the canonical forms of the substitutions of G_m. The former depend upon the characteristic determinants of the substitutions (α_{ij}), viz.:

$$\Delta(\lambda) \equiv \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22}-\lambda & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm}-\lambda \end{vmatrix}$$
$$\equiv (-1)^m (\lambda^m - a_1 \lambda^{m-1} - a_2 \lambda^{m-2} \cdots - a_m).$$

Furthermore, G_m contains a substitution in whose characteristic determinant the coefficients $a_1, a_2, \ldots, a_m$ are arbitrary marks of the $GF[p^n]$ such that $a_m \neq 0$. The required substitution is

$$(\alpha_{ij}) \equiv \begin{pmatrix} a_1 & a_2 & a_3 \cdots & a_{m-1} & a_m \\ 1 & 0 & 0 \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & 1 & 0 \end{pmatrix}.$$

The present note considers the case $m = 3$. In the article following this, Mr. Putnam treats the case $m = 4$.

Every ternary linear homogeneous substitution in the $GF[p^n]$ can be reduced by a linear ternary transformation of indices (not necessarily with coefficients in the same field) to one of the following five types of canonical forms:

$$
\begin{array}{llll}
\mathrm{A}: & x' = \lambda x, & y' = \lambda^{p^n} y, & z' = \lambda^{p^{2n}} z, \\
\mathrm{B}: & x' = \mu x, & y' = \mu^{p^n} y, & z' = \alpha z, \\
\mathrm{C}: & x' = \alpha x, & y' = \beta y \;, & z' = \gamma z, \\
\mathrm{D}: & x' = \alpha x, & y' = \beta y \;, & z' = \beta (z + y), \\
\mathrm{E}: & x' = \alpha x, & y' = \alpha (y + x), & z' = \alpha (z + y),
\end{array}
$$

where λ satisfies a cubic equation and μ a quadratic equation, each belonging to and irreducible in the $GF[p^n]$, while α, β, γ denote marks $\neq 0$ of the $GF[p^n]$.

Type A includes $\frac{1}{3}(p^{3n} - p^n)$ distinct sets of conjugate substitutions, those in different sets being not conjugate under G_3. In fact, if we replace λ by λ^{p^n} or by $\lambda^{p^{2n}}$, we obtain from A a substitution conjugate with A under G_3. Any other replacement of λ leads to a substitution not conjugate with A, since its characteristic determinant differs from that of A. Let S be a substitution of G_3 having the canonical form A, where λ is a definite mark of the $GF[p^{3n}]$ not in the $GF[p^n]$. If a substitution T of G_3 be commutative with S, and if we apply to T the same transformation of indices which reduces S to the form A, then T must take the form

$$x' = \sigma^r x, \quad y' = \sigma^{rp^n} y, \quad z' = \sigma^{rp^{2n}} z,$$

where σ is a primitive root of the $GF[p^{3n}]$ and r is some positive integer $\overline{\gtrless} p^{3n} - 1$. Hence, S is commutative with exactly $p^{3n} - 1$ substitutions of G_3, so that S is one of $N \div (p^{3n} - 1)$ conjugate substitutions within G_3, the latter having the order

$$N \equiv (p^{3n} - 1)(p^{3n} - p^n)(p^{3n} - p^{2n}).$$

The number of substitutions of G_3 reducible to the types A is, therefore,

$$\tfrac{1}{3}(p^{3n} - p^n)(p^{3n} - p^n)(p^{3n} - p^{2n}). \tag{a}$$

Type B includes $\frac{1}{2}(p^{2n} - p^n)(p^n - 1)$ distinct sets of conjugate substitutions. In fact, the replacement of μ by μ^{p^n} leads to a conjugate substitution, while any other replacement of μ or any change in α leads to a substitution not conjugate with B. A substitution of G_3 commutative with a particular substitution redu-

cible to a type B has the canonical form

$$x' = \rho^r x, \quad y' = \rho^{rp^n} y, \quad z' = \delta z,$$

where ρ is a primitive root of the $GF[p^{2n}]$ and δ belongs to the $GF[p^n]$ and r is a positive integer $\overline{\gtrless} p^{2n} - 1$. The number of such substitutions is $(p^{2n} - 1)(p^n - 1)$. Hence, the total number of substitutions of G_3 reducible to the canonical forms B is

$$\tfrac{1}{2}(p^{2n} - p^n)(p^n - 1)(p^{3n} - 1)\,p^{3n}. \tag{b}$$

Type C includes $p^n - 1$ canonical forms with $\alpha = \beta = \gamma$; $(p^n - 1)(p^n - 2)$ canonical forms with $\alpha = \beta \neq \gamma$; a like number with $\alpha = \gamma \neq \beta$; a like number with $\beta = \gamma \neq \alpha$; $(p^n - 1)(p^n - 2)(p^n - 3)$ canonical forms with α, β, γ all distinct. By a suitable transformation of indices, the multipliers α, β, γ in C are permuted in an arbitrary manner. The distinct sets of conjugate canonical substitutions C are, therefore, given by the table:

$p^n - 1$ of type C_1 with $\alpha = \beta = \gamma$;
$(p^n - 1)(p^n - 2)$ of type C_2 with only two equal multipliers, say $\alpha = \beta \neq \gamma$;
$\frac{1}{6}(p^n - 1)(p^n - 2)(p^n - 3)$ of type C_3 with α, β, γ all different.

The most general substitution of G_3 commutative with C_3 is

$$x' = ax, \quad y' = by, \quad z' = cz,$$

so that C_3 is one of $N \div (p^n - 1)^3$ conjugate substitutions within G_3. The most general substitution of G_3 commutative with C_2 is

$$x' = ax + by, \quad y' = cx + dy, \quad z' = ez,$$

so that C_2 is one of $N \div (p^{2n} - 1)(p^{2n} - p^n)(p^n - 1)$ conjugate substitutions. Finally, C_1 is commutative with every substitution of G_3 and is, therefore, conjugate only with itself. The total number of substitutions of G_3 reducible to the canonical forms C is thus:

$$(p^n - 1) + (p^{3n} - 1)(p^n - 2)\,p^{2n} + \tfrac{1}{6}(p^n - 2)(p^n - 3)(p^{3n} - 1)(p^n + 1)\,p^{3n}. \tag{c}$$

Type D includes $p^n - 1$ substitutions with $\alpha = \beta$ and $(p^n - 1)(p^n - 2)$ with $\alpha \neq \beta$, no two being conjugate under G_3.

A substitution D with $\alpha = \beta$ is commutative only with the $p^{3n}(p^n - 1)^2$ substitutions of G_3

$$x' = dy + ex, \quad y' = ay, \quad z' = by + az + cx.$$

A substitution D with $\alpha \neq \beta$ is commutative only with the $p^n(p^n-1)^2$ substitutions of G_3

$$x' = ex, \quad y' = ay, \quad z' = by + az.$$

The number of substitutions of G_3 reducible to the types D is thus:

$$(p^n-1)(p^{3n}-1)(p^n+1) + (p^n-1)(p^n-2)(p^{3n}-1)(p^n+1)\,p^{2n}. \qquad \text{(d)}$$

Type E includes $p^n - 1$ substitutions, no two of which are conjugate under G_3. Each is commutative only with the $p^{2n}(p^n-1)$ substitutions of G_3,

$$x' = ax, \quad y' = bx + ay, \quad z' = cx + by + az.$$

The number of substitutions reducible to the types E is thus:

$$(p^n-1)(p^{3n}-1)(p^{2n}-1)\,p^n. \qquad \text{(e)}$$

A check upon the above enumeration of the substitutions G_3 consists in verifying that the sum of the numbers (a), (b), (c), (d), (e) equals the order N of G_3.

THE UNIVERSITY OF TEXAS, *May*, 1900.

Distribution of the Quaternary Linear Homogeneous Substitutions in a Galois Field into Complete Sets of Conjugate Substitutions.

By T. M. Putnam.

The classification used is based upon the canonical forms of linear homogeneous substitutions in an arbitrary Galois field.*

The homogeneous substitution

$$x' = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 w, \quad y' = x, \quad z' = y, \quad w' = z$$

has for its characteristic determinant

$$\Delta(\lambda) \equiv \lambda^4 - \alpha_1 \lambda^3 - \alpha_2 \lambda^2 - \alpha_3 \lambda - \alpha_4,$$

where α_1, α_2, α_3, α_4 may be arbitrary marks in the $GF[p^n]$ such that $\alpha_4 \neq 0$. Hence, substitutions exist for which $\Delta(\lambda)$ in the $GF[p^n]$ is irreducible; the product of a linear factor and an irreducible cubic; the product of two distinct irreducible quadratics; the square of an irreducible quadratic; the product of an irreducible quadratic and two linear factors distinct or equal; finally, the product of four linear factors, some or all of which may be equal.

Type I. If the characteristic determinant is irreducible, the substitution may be reduced to the canonical form

$$x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \lambda^{p^{2n}} z, \quad w' = \lambda^{p^{3n}} w.$$

where λ is an arbitrary mark in the $GF[p^{4n}]$ but not in the $GF[p^{2n}]$. There are then $p^{4n} - p^{2n}$ ways of setting up this canonical form. But replacing λ by λ^{p^n}, $\lambda^{p^{2n}}$, or $\lambda^{p^{3n}}$, we obtain a substitution conjugate with the original. This can

* Dr. L. E. Dickson, Amer. Jour. of Math., vol. XXII, pp. 121-137.

happen by no other replacement; hence, there are $\frac{p^{4n}-p^{2n}}{4}$ distinct sets of conjugate substitutions. The most general substitution commutative with one of this type is

$$x' = \mu x, \quad y' = \mu^{p^n} y, \quad z' = \mu^{p^{2n}} z, \quad w' = \mu^{p^{3n}} w,$$

where μ is an arbitrary mark $\neq 0$ in the $GF[p^{4n}]$. There are then $p^{4n}-1$ substitutions commutative with each one of this type. Hence, each set contains $\frac{N}{p^{4n}-1}$ conjugate substitutions, N being the order of the group, viz., $(p^{4n}-1)(p^{4n}-p^n)(p^{4n}-p^{2n})(p^{4n}-p^{3n})$. The total number of substitutions, then, that can be reduced to this canonical form is $\frac{(p^{4n}-p^{2n})N}{4(p^{4n}-1)}$. The period of any one of them is evidently a factor of $p^{4n}-1$ but not of $p^{2n}-1$.

Type II. If the characteristic determinant is the product of an irreducible cubic and a linear factor, the canonical form becomes

$$x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \lambda^{p^{2n}} z, \quad w' = \alpha w,$$

where λ is an arbitrary mark in the $GF[p^{3n}]$, but not in the $GF[p^n]$, and α is arbitrary in the $GF[p^n]$. The period of a substitution of this type will be a factor of $p^{3n}-1$, but not of p^n-1. λ can take $p^{3n}-p^n$ values and α can take p^n-1, but replacing λ by λ^{p^n} or $\lambda^{p^{2n}}$, we obtain substitutions conjugate with the original. Hence, there are $\frac{(p^{3n}-p^n)(p^n-1)}{3}$ distinct sets of conjugate substitutions. The general substitution commutative with one of this type has the form

$$x' = \mu x, \quad y' = \mu^{p^n} y, \quad z' = \mu^{p^{2n}} z, \quad w' = \beta w,$$

where μ is arbitrary in the $GF[p^{3n}]$ and β in the $GF[p^n]$, in all, then, $(p^{3n}-1)(p^n-1)$. Hence, there are $\frac{N}{(p^{3n}-1)(p^n-1)}$ substitutions in each of the conjugate sets, giving in all $\frac{(p^{3n}-p^n)N}{3(p^{3n}-1)}$ substitutions reducible to this canonical form.

Type III. When the characteristic determinant is the product of two dis-

tinct quadratics, the canonical form becomes

$$x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \mu z, \quad w' = \mu^{p^n} w,$$

where λ and μ are any marks in the $GF[p^{2n}]$ not in the $GF[p^n]$ and $\mu \neq \lambda$ and $\mu \neq \lambda^{p^n}$. There are evidently $(p^{2n} - p^n)(p^{2n} - p^n - 2)$ ways of setting up this canonical form; for to each of the $p^{2n} - p^n$ values of λ, μ takes $p^{2n} - p^n - 2$ values not equal to λ or λ^{p^n}. But the substitutions with the multipliers $(\lambda, \lambda^{p^n}, \mu, \mu^{p^n})$, $(\lambda^{p^n}, \lambda, \mu, \mu^{p^n})$, $(\lambda, \lambda^{p^n}, \mu^{p^n}, \mu)$, $(\lambda^{p^n}, \lambda, \mu^{p^n}, \mu)$ and the four others with the λ's and μ's interchanged are conjugates. Hence, there are $\frac{1}{8}(p^{2n} - p^n)(p^{2n} - p^n - 2)$ distinct sets of conjugate substitutions. The form of the commutative substitution here is

$$x' = \sigma x, \quad y' = \sigma^{p^n} y, \quad z' = v z, \quad w' = v^{p^n} w, \qquad (\sigma,\ v \text{ arb. in } GF[p^{2n}]).$$

The total number of commutative substitutions is then $(p^{2n} - 1)^2$, and hence each set has $\dfrac{N}{(p^{2n}-1)^2}$ substitutions. In all there are, then, $\dfrac{Np^n(p^n - 2)}{8(p^{2n} - 1)}$ substitutions of this type, each of period a factor of $p^{2n} - 1$ but not of $p^n - 1$.

Type IV. The characteristic equation for this type is the square of an irreducible quadratic. Two canonical forms occur,

$$\begin{array}{llll} (1) \quad x' = \lambda x, & y' = \lambda(y + x), & z' = \lambda^{p^n} z, & w' = \lambda^{p^n}(w + z); \\ (2) \quad x' = \lambda x, & y' = \lambda y, & z' = \lambda^{p^n} z, & w' = \lambda^{p^n} w, \end{array}$$

where λ is arbitrary in the $GF[p^{2n}]$, but is not in the $GF[p^n]$. There will be $\dfrac{p^{2n} - p^n}{2}$ distinct sets of conjugate substitutions in each case. The corresponding commutative substitutions are

$$\text{(i)} \quad x' = \mu x, \quad y' = \sigma x + \mu y, \quad z' = \mu^{p^n} z, \quad w' = \sigma^{p^n} z + \mu^{p^n} w,$$

μ and σ being arbitrary in the $GF[p^{2n}]$.

$$\text{(ii)} \quad x' = \sigma_1 x + \mu_1 y, \quad y' = \sigma_2 x + \mu_2 y, \quad z' = \sigma_1^{p^n} z + \mu_1^{p^n} w, \quad w' = \sigma_2^{p^n} z + \mu_2^{p^n} w,$$

where σ_1, σ_2, μ_1, μ_2 are arbitrary marks of the $GF[p^{2n}]$. There are $p^{2n}(p^{2n} - 1)$ of form (i) and $(p^{4n} - 1)(p^{4n} - p^{2n})$ of form (ii), the latter being the number of ways that the determinant of the substitution (ii), viz., $(\sigma_1 \mu_2 - \mu_1 \sigma_2)^{p^n + 1}$ can be set up

with $\sigma_1, \sigma_2, \mu_1, \mu_2$ arbitrary marks in the $GF[p^{2n}]$. Hence, from (1) there are in all $\dfrac{N}{2p^n(p^n+1)}$, and from (2) $\dfrac{N}{2p^n(p^n+1)(p^{4n}-1)}$ substitutions, those of (1) being of period a factor of $p(p^{2n}-1)$ but not of $p(p^n-1)$, and those of (2) of period a factor of $p^{2n}-1$ but not of p^n-1.

Type V. When $\Delta(\lambda)$ is the product of an irreducible quadratic and two distinct linear factors, the canonical form is

$$x'=\alpha x, \quad y'=\beta y, \quad z'=\lambda z, \quad w'=\lambda^{p^n}w,$$

where α and β are arbitrary in the $GF[p^n]$, $\alpha \neq \beta$, and λ is arbitrary in the $GF[p^{2n}]$ but not in the $GF[p^n]$. If α and β, or λ and λ^{p^n} are interchanged. conjugate substitutions are obtained; hence, there are $\dfrac{(p^{2n}-p^n)(p^n-1)(p^n-2)}{4}$ distinct sets of conjugate substitutions. The commutative substitutions are of the form

$$x'=ax, \quad y'=by, \quad z'=\mu z, \quad w'=\mu^{p^n}w,$$

with a and b arbitrary in the $GF[p^n]$, and μ arbitrary in the $GF[p^{2n}]$, in all, therefore, $(p^{2n}-1)(p^n-1)^2$. Hence there are from this type $\dfrac{p^n(p^n-2)N}{4(p^{2n}-1)}$ substitutions of period a factor of $p^{2n}-1$ but not of p^n-1.

Type VI. $\Delta(\lambda)$ for this type is the same as in Type V with the two linear factors coincident. Two canonical forms occur,

$$\begin{array}{llll} (1) \quad x'=\alpha x, & y'=\alpha(y+x), & z'=\lambda z, & w'=\lambda^{p}w; \\ (2) \quad x'=\alpha x, & y'=\alpha y, & z'=\lambda z, & w'=\lambda^{p^n}w. \end{array}$$

(1) is of period a factor of $p(p^{2n}-1)$ but not of $p(p^n-1)$, and (2) is of period a factor of $(p^{2n}-1)$ but not of p^n-1. In each case there are $\dfrac{(p^{2n}-p^n)(p^n-1)}{2}$ distinct sets of conjugate substitutions. The commutative substitutions have the respective forms,

$$\begin{array}{llll} \text{(i)} \quad x'=ax, & y'=by+ax, & z'=\mu z, & w'=\mu^{p^n}w; \\ \text{(ii)} \quad x'=ax+by, & y'=cx+dy, & z'=\mu z, & w'=\mu^{p^n}w. \end{array}$$

There are $p^n(p^n-1)(p^{2n}-1)$ of form (i) and $(p^{2n}-1)^2(p^{2n}-p^n)$ of form (ii).

Hence, belonging to the two canonical forms, there are respectively $\frac{N}{2(p^n+1)}$ and $\frac{N}{2(p^n+1)(p^{2n}-1)}$ substitutions.

Type VII. In this case $\Delta(\lambda)$ is the product of four distinct linear factors. The canonical form is

$$x' = \alpha x, \quad y' = \beta y, \quad z' = \gamma z, \quad w' = \delta w.$$

The determinant of this substitution is $\Delta = \alpha\beta\gamma\delta$, with $(p^n-1)^4$ sets of solutions. But three multipliers are equal in $4(p^n-1)(p^n-2)$ of the sets [see IX]; two only will be equal for $6(p^n-1)(p^n-2)(p^n-3)$ others [see VIII]; they will be equal in pairs in $3(p^n-1)(p^n-2)$ of the sets [XI], while for p^n-1 all four will be equal [X]. Excluding these, there remain $(p^n-1)(p^n-2)(p^n-3)(p^n-4)$ sets of distinct multipliers conjugate, however, in sets of $4!$. A substitution of this type is commutative with the $(p^n-1)^4$ substitutions of the form

$$x' = ax, \quad y' = by, \quad z' = cz, \quad w' = dw.$$

Hence there are $\frac{1}{24}(p^n-1)(p^n-2)(p^n-3)(p^n-4)$ sets with $\frac{N}{(p^n-1)^4}$ substitutions in each; in all, therefore, there are $\frac{(p^n-2)(p^n-3)(p^n-4)N}{24(p^n-1)^3}$ substitutions of period a factor of p^n-1 from this type.

Type VIII. For this type just two of the linear factors of $\Delta(\lambda)$ are equal, the canonical forms being

$$\begin{aligned}
&(1) \quad x' = \alpha x, \quad y' = \beta y, \quad z' = \gamma z, \quad w' = \gamma(w+z);\\
&(2) \quad x' = \alpha x, \quad y' = \beta y, \quad z' = \gamma z, \quad w' = \gamma w,
\end{aligned}$$

(1) is of period a factor of $p(p^n-1)$ and (2) is of period a factor of p^n-1. For each there are $\frac{(p^n-1)(p^n-2)(p^n-3)}{2}$ distinct sets of conjugate substitutions. The $p^n(p^n-1)^3$ substitutions of the form

$$x' = ax, \quad y' = by, \quad z' = cz, \quad w' = dz + cw$$

are commutative with (1), while there are $(p^n-1)^2(p^{2n}-1)(p^{2n}-p^n)$ commutative with (2), viz., all those of the form

$$x'=ax,\quad y'=by,\quad z'=cz+dw,\quad w'=c'z+d'w.$$

Therefore, there are $\dfrac{(p^n-2)(p^n-3)N}{2p^n(p^n-1)^2}$ substitutions of type (1) and $\dfrac{(p^n-2)(p^n-3)N}{2(p^n-1)^3(p^n+1)p^n}$ of type (2).

Type IX. In this case there are three of the linear factors of $\Delta(\lambda)$ equal, giving rise to three canonical forms,

$$\begin{array}{llll} (1)\ x'=\alpha x, & y'=\beta y, & z'=\beta(z+y), & w'=\beta(w+z), \\ (2)\ x'=\alpha x, & y'=\beta y, & z'=\beta z, & w'=\beta(w+z), \\ (3)\ x'=\alpha x, & y'=\beta y, & z'=\beta z, & w'=\beta w. \end{array}$$

(1) and (2) are of periods factors of $p(p^n-1)$ (if $p=2$, (1) is of period a factor of $4(p^n-1)$), (3) is of period a factor of p^n-1. There are just $(p^n-1)(p^n-2)$ sets of conjugate substitutions for each subtype.

The respective substitutions commutative with (1), (2) and (3) have the forms

$$\begin{array}{llll} \text{(i)} & x'=ax, & y'=by, & z'=bz+cy,\quad w'=bw+cz+dy; \\ \text{(ii)} & x'=ax, & y'=by+cz, & z'=dz,\quad w'=ey+fz+dw; \\ \text{(iii)} & x'=dx, & y'=a_1y+b_1z+c_1w, & z'=a_2y+b_2z+c_2w, \\ & & & w'=a_3y+b_3z+c_3w. \end{array}$$

There are $p^{2n}(p^n-1)^2$ of form (i), $p^{3n}(p^n-1)^3$ of form (ii), and $(p^{3n}-1)(p^{3n}-p^n)(p^{3n}-p^{2n})(p^n-1)$ of form (iii). There will be then in all $\dfrac{(p^n-2)N}{p^{2n}(p^n-1)}$ substitutions from the subtype (1), $\dfrac{(p^n-2)N}{p^{3n}(p^n-1)^2}$ from (2), and $\dfrac{(p^n-2)N}{(p^{3n}-1)(p^{3n}-p^n)(p^{3n}-p^{2n})}$ from (3).

Type X. When $\Delta(\lambda)$ is the fourth power of a single linear factor, five canonical forms arise,

$$\begin{array}{lllll} (1) & x'=\alpha x, & y'=\alpha(y+x), & z'=\alpha(z+y), & w'=\alpha(w+z); \\ (2) & x'=\alpha x, & y'=\alpha y, & z'=\alpha(z+y), & w'=\alpha(w+z); \\ (3) & x'=\alpha x, & y'=\alpha(y+x), & z'=\alpha z, & w'=\alpha(w+z); \\ (4) & x'=\alpha x, & y'=\alpha y, & z'=\alpha z, & w'=\alpha(w+z); \\ (5) & x'=\alpha x, & y'=\alpha y, & z'=\alpha z, & w'=\alpha w. \end{array}$$

If $p = 2$, (1) and (2) are of period $4(p^n - 1)$ or a factor of it; if $p = 3$, (1) is of period a factor of $9(p^n - 1)$; for $p > 3$, the period of (1) is a factor of $p(p^n - 1)$; for $p > 2$, the period of (2) is a factor of $p(p^n - 1)$. (3) and (4) are always of period a factor of $p(p^n - 1)$, and (5) is of period a factor of $p^n - 1$.

The substitutions commutative with the above have the respective forms

(i) $x' = ax, \quad y' = bx + ay, \quad z' = cx + by + az, \quad w' = dx + cy + bz + aw;$

(ii) $x' = ax + by, \quad y' = cy, \quad z' = dy + cz, \quad w' = ex + fy + dz + cw;$

(iii) $x' = a_1x + b_1y, \quad y' = a_2x + a_1y + b_2z + b_1w, \quad z' = a_3x + b_3z, \quad w' = a_4x + a_3y + b_4z + b_3w;$

(iv) $x' = ax + by + cz, \quad y' = a_1x + b_1y + c_1z, \quad z' = c_2z, \quad w' = a_4x + b_4y + c_4z + c_2w;$

(v) is commutative with every substitution of the group.

The number of substitutions in each of these types is

(1) $p^{3n}(p^n - 1)$, (2) $p^{4n}(p^n - 1)^2$,
(3) $p^{4n}(p^{2n} - 1)(p^{2n} - p^n)$, (4) $p^{5n}(p^n - 1)(p^{2n} - 1)(p^{2n} - p^n)$.

Hence the totals for the respective subtypes are

(1) $\dfrac{N}{p^{3n}}$, (2) $\dfrac{N}{p^{4n}(p^n - 1)}$, (3) $\dfrac{N}{p^{5n}(p^{2n} - 1)}$, (4) $\dfrac{N}{p^{6n}(p^{2n} - 1)(p^n - 1)}$,

and, finally (5) with $p^n - 1$ substitutions.

Type XI. When the multipliers are equal in pairs, three canonical forms arise,

(1) $x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \beta z, \quad w' = \beta(w + z);$
(2) $x' = \alpha x, \quad y' = \alpha y, \quad z' = \beta z, \quad w' = \beta(w + z);$
(3) $x' = \alpha x, \quad y' = \alpha y, \quad z' = \beta z, \quad w' = \beta w.$

There will be $\dfrac{(p^n - 1)(p^n - 2)}{2}$ distinct sets of conjugate substitutions for (1) and (3), since interchanging α and β gives conjugate substitutions. This is not true, however, for (2), which has, therefore, $(p^n - 1)(p^n - 2)$ distinct sets.

The substitutions commutative with these three subtypes are, respectively,

$$\begin{array}{lllll}
\text{(i)} & x' = ax, & y' = bx + ay, & z' = cz, & w' = dz + cw\,; \\
\text{(ii)} & x' = ax + by, & y' = cx + dy, & z' = ez, & w' = fz + ew, \\
\text{(iii)} & x' = ax + by, & y' = a_1x + b_1y, & z' = cz + dw, & w' = c_1z + d_1w.
\end{array}$$

There will be $p^{2n}(p^n - 1)^2$ substitutions of form (i), $p^{2n}(p^n - 1)^3(p^n + 1)$ of form (ii), and $(p^{2n} - 1)^2(p^{2n} - p^n)^2$ of form (iii). The total number of substitutions of each type is, therefore,

$$(1)\ \frac{(p^n - 2)\,N}{2\,(p^n - 1)\,p^{2n}}, \quad (2)\ \frac{(p^n - 2)\,N}{p^{2n}(p^n - 1)^2(p^n + 1)}, \quad (3)\ \frac{(p^n - 2)\,N}{2\,(p^{2n} - p^n)^2(p^n + 1)(p^{2n} - 1)}.$$

As a check on the above results, the sum of all the totals for the various canonical forms will be found to equal N, the order of the group. For $p^n = 2$ the group is simply isomorphic with the alternating group on eight letters, and the above results also agree with those for that group.

UNIVERSITY OF TEXAS, *May*, 1900.

On the Determination and Solution of the Metacyclic Quintic Equations with Rational Coefficients.

By J. C. Glashan.

(The following paper is in tardy fulfillment of a promise made to the late Professor George Paxton Young. See *American Journal of Mathematics*, vol. VI, page 114).

§1. If $c\ d\ e\ f$ are given rational numbers, and if

$$\begin{aligned}
A &\equiv 3c^2 + e,\\
B &\equiv 15c^4 - 2c^2e + 8cd^2 - 2df + 3e^2,\\
C &\equiv 25c^6 - 35c^4e + 40c^3d^2 + 2c^2df + 11c^2e^2 - 28cd^2e - cf^2 + 16d^4 + 2def - e^3,\\
D &\equiv 3456c^5f^2 - 11520c^4def + 6400c^4e^3 + 5120c^3d^3f - 3200c^3d^2e^2 - 1440c^3ef^2\\
&\quad + 2640c^2d^2f^2 + 4480c^2de^2f - 2560c^2e^4 - 10080cd^3ef + 5760cd^2e^3 - 120cdf^3\\
&\quad + 160ce^2f^2 + 3456d^5f - 2160d^4e^2 + 360d^2ef^2 - 640de^3f + 256e^5 + f^4,
\end{aligned}$$

the quintic equation

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0 \tag{1}$$

will be solvable by radicals if the sextic equation

$$(u^3 - 5Au^2 + 5Bu - 5C)^2 - Du = 0 \tag{2}$$

have a rational root, but if the sextic (2) have not a rational root, the quintic (1) will not be solvable by radicals. (In examining equation (2) for a rational root, only common factors of C and D of the form $m^2 + n^2$ or the form m^2, need be tried as values of u.)

If the sextic (2) have a rational root, denote it by u_1, and let

$$E \equiv 2cd + f,$$
$$F \equiv 10c^3d + 11c^2f - 18cde + 12d^3 + ef,$$
$$G \equiv 50c^5d - 59c^4f + 20c^3de + 40c^2d^3 + 42c^2ef - 48cd^2f - 38cde^2 + 44d^3e - df^2 + e^2f,$$
$$\tau_2 = \tfrac{1}{5}\sqrt{u_1}.$$
$$\tau_3 = \frac{(Eu_1^2 - 2Fu_1 + G)\,\tau_2}{u_1^3 - 3Au_1^2 + Bu_1 + C},$$
$$\alpha_1 = \tfrac{1}{2}\left[-(d-\tau_3) + \sqrt{\{(d-\tau_3)^2 + 4(c-\tau_2)(c^2-\tau_2^2)\}}\right],$$
$$\alpha_2 = \tfrac{1}{2}\left[-(d+\tau_3) - \sqrt{\{(d+\tau_3)^2 + 4(c+\tau_2)(c^2-\tau_2^2)\}}\right],$$
$$\alpha_3 = \tfrac{1}{2}\left[-(d+\tau_3) + \sqrt{\{(d+\tau_3)^2 + 4(c+\tau_2)(c^2-\tau_2^2)\}}\right],$$
$$\alpha_4 = \tfrac{1}{2}\left[-(d-\tau_3) - \sqrt{\{(d-\tau_3)^2 + 4(c-\tau_2)(c^2-\tau_2^2)\}}\right].$$

and
$$\omega^5 - 1 = 0,$$
then will
$$x = \omega\left\{\frac{\alpha_1^2\alpha_3}{(c+\tau_2)^2}\right\}^{\frac{1}{5}} + \omega^2\left\{\frac{\alpha_3^2\alpha_4}{(c-\tau_2)^2}\right\}^{\frac{1}{5}} + \omega^3\left\{\frac{\alpha_2^2\alpha_1}{(c-\tau_2)^2}\right\}^{\frac{1}{5}} + \omega^4\left\{\frac{\alpha_4^2\alpha_2}{(c+\tau_2)^2}\right\}^{\frac{1}{5}}.$$

§2. The preceding propositions may be proved as follows:

Let
$$x = \omega y_1 + \omega^2 y_2 + \omega^3 y_3 + \omega^4 y_4,$$
in which
$$\omega^5 - 1 = 0,$$
then will
$$x^5 - 10\sigma_2 x^3 - 10\sigma_3 x^2 - 5(2\sigma_4 - \sigma_2^2 - 3\tau_2^2)\,x - 2\sigma_5 + 20\tau_2\tau_3 = 0, \qquad (3)$$
in which

$$2\sigma_2 = y_1y_4 + y_2y_3, \qquad 2\tau_2 = y_1y_4 - y_2y_3,$$
$$2\sigma_3 = y_1^2y_3 + y_2^2y_1 + y_3^2y_4 + y_4^2y_2, \qquad 2\tau_3 = y_1^2y_3 - y_2^2y_1 - y_3^2y_4 + y_4^2y_2,$$
$$2\sigma_4 = y_1^3y_2 + y_2^3y_4 + y_3^3y_1 + y_4^3y_3,$$
$$2\sigma_5 = y_1^5 + y_2^5 + y_3^5 + y_4^5.$$

Comparing equations (1) and (3), gives us at once

$$\sigma_2 = -c,$$
$$\sigma_3 = -d,$$
$$2\sigma_4 - \sigma_2^2 - 3\tau_2^2 = -e,$$
$$2\sigma_5 - 20\tau_2\tau_3 = -f.$$

To these we add

$$\begin{aligned}
\sigma_3^2-\tau_3^2 &= 2(\sigma_2\sigma_4-\tau_2\tau_4),\\
\sigma_4^2-\tau_4^2 &= 2\sigma_2(\sigma_3^2+\tau_3^2)-4\tau_2\sigma_3\tau_3-4(\sigma_2^2-\tau_2^2)^2,\\
(\sigma_3+\tau_3)(\sigma_4+\tau_4) &= (\sigma_2-\tau_2)(\sigma_5+\tau_5)+(\sigma_2+\tau_2)^2(\sigma_3-\tau_3),\\
(\sigma_3-\tau_3)(\sigma_4-\tau_4) &= (\sigma_2+\tau_2)(\sigma_5-\tau_5)+(\sigma_2-\tau_2)^2(\sigma_3+\tau_3),
\end{aligned}$$

in which

$$\begin{aligned}
2\tau_4 &= y_1^3y_2-y_2^3y_4-y_3^3y_1+y_4^3y_3,\\
2\tau_5 &= y_1^5-y_2^5-y_3^5+y_4^5.
\end{aligned}$$

On eliminating σ_2, σ_3, σ_4, σ_5, τ_3, τ_4 and τ_5 from these equations and writing u for $25\tau_2^2$, we obtain

$$\begin{aligned}
[\{&u^3-3(3c^2+e)u^2+(15c^4-2c^2e+8cd^2-2df+3e^2)u\\
&+25c^6-35c^4e+40c^3d^2+2c^2df+11c^2e^2-28cd^2e-cf^2\\
&+16d^4+2def-e^3\}\times\\
\{&cu^4-(20c^3+8ce-d^2)u^3+(150c^5+20c^3e+55c^2d^2-14cdf\\
&+38ce^2-7d^2e+f^2)u^2-(500c^7-400c^5e+625c^4d^2+36c^3df\\
&+100c^3e^2-270c^2d^2e+24c^2f^2+200cd^4-104cdef+56ce^3\\
&+66d^3f-31d^2e^2+ef^2)u+25(c^3-ce+d^2)(25c^6-35c^4e+40c^3d^2\\
&+2c^2df+11c^2e^2-28cd^2e-cf^2+16d^4+2def-e^3)\}\\
&-\{(2cd+f)u^2-2(10c^3d+11c^2f-18cde+12d^3+ef)u+50c^5d-59c^4f\\
&+20c^3de+40c^2d^3+42c^2ef-48cd^2f-38cde^2+44d^3e-df^2+e^2f\}^2u]\\
&\div(cu+c^3-ce+d^2)=0.
\end{aligned}$$

On performing the indicated multiplications and division, the resulting sextic equation may easily be reduced to the form in which it has been given in equation (2) of §1, viz.:

$$(u^3-5Au^2+5Bu-5C)^2-Du=0.$$

In the course of the elimination, the equation

$$\tau_3=\frac{(Eu^2-2Fu+G)\tau_2}{u^3-3Au^2+Bu+C}$$

is obtained, and thus τ_3 is given in terms of τ_2.

If τ_2 and τ_3 have been determined, we have

$$y_1 y_4 = \sigma_2 + \tau_2 = -(c - \tau_2),$$
$$y_2 y_3 = \sigma_2 - \tau_2 = -(c + \tau_2),$$
$$y_1^2 y_3 + y_4^2 y_2 = \sigma_3 + \tau_3 = -(d - \tau_3),$$
$$y_2^2 y_1 + y_3^2 y_4 = \sigma_3 - \tau_3 = -(d + \tau_3),$$

$$\therefore \quad y_1^2 y_3 - y_4^2 y_2 = \sqrt{\{(d-\tau_3)^2 + 4(c-\tau_2)^2(c+\tau_2)\}},$$
$$y_2^2 y_1 - y_3^2 y_4 = \sqrt{\{(d+\tau_3)^2 + 4(c+\tau_2)^2(c-\tau_2)\}},$$

$$\therefore \quad y_1^2 y_3 = \tfrac{1}{2}[-(d-\tau_3) + \sqrt{\{(d-\tau_3)^2 + 4(c-\tau_2)(c^2-\tau_2^2)\}}] = \alpha_1,$$
$$y_4^2 y_2 = \tfrac{1}{2}[-(d-\tau_3) - \sqrt{\{(d-\tau_3)^2 + 4(c-\tau_2)(c^2-\tau_2^2)\}}] = \alpha_4,$$
$$y_2^2 y_1 = \tfrac{1}{2}[-(d+\tau_3) + \sqrt{\{(d+\tau_3)^2 + 4(c+\tau_2)(c^2-\tau_2^2)\}}] = \alpha_3,$$
$$y_3^2 y_4 = \tfrac{1}{2}[-(d+\tau_3) - \sqrt{\{(d+\tau_3)^2 + 4(c+\tau_2)(c^2-\tau_2^2)\}}] = \alpha_2,$$

$$\therefore \quad y_1^5 = \frac{\alpha_1^2 \alpha_3}{(c+\tau_2)^2}, \qquad y_4^5 = \frac{\alpha_4^2 \alpha_2}{(c+\tau_2)^2},$$
$$y_2^5 = \frac{\alpha_3^2 \alpha_4}{(c-\tau_2)^2}, \qquad y_3^5 = \frac{\alpha_2^2 \alpha_1}{(c-\tau_2)^2},$$

and $$x = \omega y_1 + \omega^2 y_2 + \omega^3 y_3 + \omega^4 y_4.$$

EXAMPLES.

1. $$x^5 + 3x^2 + 2x - 1 = 0,$$

$$\therefore \quad c = 0, \quad d = .3, \quad e = .4, \quad f = -1,$$

$\therefore$ the critical sextic is

$$(u^3 - 2u^2 + 5.4u + .872)^2 - 17.672u = 0,$$
$$\therefore \quad u_1 = .2,$$
$$\therefore \quad \tau_2 = \tfrac{1}{25}\sqrt{5} \text{ and } \tau_3 = \tfrac{7}{50}\sqrt{5},$$
$$\therefore \quad \alpha_1 = \frac{1}{500}[-75 + 35\sqrt{5} + \sqrt{\{470(25 - 11\sqrt{5})\}}],$$
$$\alpha_2 = \frac{1}{500}[-75 - 35\sqrt{5} - \sqrt{\{470(25 + 11\sqrt{5})\}}],$$
$$\alpha_3 = \frac{1}{500}[-75 - 35\sqrt{5} + \sqrt{\{470(25 + 11\sqrt{5})\}}],$$
$$\alpha_4 = \frac{1}{500}[-75 + 35\sqrt{5} - \sqrt{\{470(25 - 11\sqrt{5})\}}],$$

$$\therefore \quad y_1^5 = \frac{13}{500}(15 - 7\sqrt{5}) + \sqrt{\left\{\left(\frac{13}{500}\right)^2 (15 - 7\sqrt{5})^2 - \left(\frac{\sqrt{5}}{25}\right)^5\right\}},$$

$$y_2^5 = \frac{13}{500}(15 + 7\sqrt{5}) - \sqrt{\left\{\left(\frac{13}{500}\right)^2 (15 + 7\sqrt{5})^2 + \left(\frac{\sqrt{5}}{25}\right)^5\right\}},$$

$$y_3^5 = \frac{13}{500}(15 + 7\sqrt{5}) + \sqrt{\left\{\left(\frac{13}{500}\right)^2 (15 + 7\sqrt{5})^2 + \left(\frac{\sqrt{5}}{25}\right)^5\right\}},$$

$$y_4^5 = \frac{13}{500}(15 - 7\sqrt{5}) - \sqrt{\left\{\left(\frac{13}{500}\right)^2 (15 - 7\sqrt{5})^2 - \left(\frac{\sqrt{5}}{25}\right)^5\right\}},$$

and $$x = \omega y_1 + \omega^2 y_2 + \omega^3 y_3 + \omega^4 y_4.$$

The value of y_1^5 may also be written

$$y_1^5 = \frac{13}{500}(15 - 7\sqrt{5}) + \frac{1}{2500}\sqrt{\{94(21125 - 9439\sqrt{5})\}},$$

with corresponding values of y_2^5, y_3^5, y_4^5.

(This is the resolvent quintic of the modular equation of the 47[th] order. It was solved by Professor G. P. Young in the *American Journal of Mathematics*, vol. X, pp. 108–110.)

2. $$x^5 - 10x^3 - 20x^2 - 1505x - 7412 = 0,$$

$$\therefore \quad c = -1, \quad d = -2, \quad e = -301, \quad f = -7412,$$

$\therefore$ the critical sextic is

$$(u^3 + 1490u^2 + 1213700u - 371439000)^2 - 1883801304320000u = 0,$$

$$\therefore \quad u_1 = 50,$$

$$\therefore \quad \tau_2 = -\sqrt{2} \quad \text{and} \quad \tau_3 = 8\sqrt{2},$$

$$\therefore \quad \alpha_1 = 1 + 4\sqrt{2} + \sqrt{(34 + 7\sqrt{2})},$$

$$\alpha_2 = 1 - 4\sqrt{2} - \sqrt{(34 - 7\sqrt{2})},$$

$$\alpha_3 = 1 - 4\sqrt{2} + \sqrt{(34 - 7\sqrt{2})},$$

$$\alpha_4 = 1 + 4\sqrt{2} - \sqrt{(34 + 7\sqrt{2})},$$

$$\therefore \quad y_1^5 = 9(197 - 139\sqrt{2}) + \sqrt{\{81(197 - 139\sqrt{2})^2 - (1 - \sqrt{2})^5\}},$$

$$y_2^5 = 9(197 + 139\sqrt{2}) - \sqrt{\{81(197 + 139\sqrt{2})^2 - (1 + \sqrt{2})^5\}},$$

$$y_3^5 = 9(197 + 139\sqrt{2}) + \sqrt{\{81(197 + 139\sqrt{2})^2 - (1 + \sqrt{2})^5\}},$$

$$y_4^5 = 9(197 - 139\sqrt{2}) - \sqrt{\{81(197 - 139\sqrt{2})^2 - (1 - \sqrt{2})^5\}},$$

and $$x = \omega y_1 + \omega^2 y_2 + \omega^3 y_3 + \omega_4 y_4.$$

(This equation was solved by Professor G. P. Young in the *American Journal of Mathematics*, vol. X, pp. 115, 116.)

3. $$x^5 - 4x^4 + 4x^3 - 5x^2 + 12x - 1 = 0.$$

Let $$x = \tfrac{1}{5}(z + 4),$$

$\therefore$ $$z^5 - 60z^3 - 705z^2 + 3460z + 19179 = 0,$$

$\therefore$ $$c = -6, \quad d = -70.5, \quad e = 692, \quad f = 19179,$$

$\therefore$ the critical sextic is

$$(u^3 - 4000u^2 + 19359375u - 4982421875)^2 - 5504299926757 8125u = 0,$$

$\therefore$ $$u_1 = 125,$$

$\therefore$ $$\tau_2 = \sqrt{5}, \qquad \tau_3 = \tfrac{83}{2}\sqrt{5},$$

$\therefore$
$$\alpha_1 = \tfrac{1}{4}\{141 + 83\sqrt{5} + \sqrt{(51350 + 22910\sqrt{5})}\},$$
$$\alpha_2 = \tfrac{1}{4}\{141 - 83\sqrt{5} - \sqrt{(51350 - 22910\sqrt{5})}\},$$
$$\alpha_3 = \tfrac{1}{4}\{141 - 83\sqrt{5} + \sqrt{(51350 - 22910\sqrt{5})}\},$$
$$\alpha_4 = \tfrac{1}{4}\{141 + 83\sqrt{5} - \sqrt{(51350 + 22910\sqrt{5})}\},$$

$\therefore$
$$y_1^5 = -\tfrac{1}{4}(15029 + 7135\sqrt{5}) - \sqrt{\{\tfrac{1}{16}(15029 + 7135\sqrt{5})^2 - (6 + \sqrt{5})^5\}},$$
$$y_2^5 = -\tfrac{1}{4}(15029 - 7135\sqrt{5}) - \sqrt{\{\tfrac{1}{16}(15029 - 7135\sqrt{5})^2 - (6 - \sqrt{5})^5\}},$$
$$y_3^5 = -\tfrac{1}{4}(15029 - 7135\sqrt{5}) + \sqrt{\{\tfrac{1}{16}(15029 - 7135\sqrt{5})^2 - (6 - \sqrt{5})^5\}},$$
$$y_4^5 = -\tfrac{1}{4}(15029 + 7135\sqrt{5}) + \sqrt{\{\tfrac{1}{16}(15029 + 7135\sqrt{5})^2 - (6 + \sqrt{5})^5\}},$$

and

$$x = \tfrac{1}{5}(4 + \omega y_1 + \omega^2 y_2 + \omega^3 y_3 + \omega^4 y_4).$$

The value of y_1^5 may also be written

$$y_1^5 = -\tfrac{1}{4}(15029 + 7135\sqrt{5}) - \tfrac{15}{4}\sqrt{\{158(13505 + 6029\sqrt{5})\}},$$

with corresponding values of y_2^5, y_3^5, y_4^5.

(This is the resolvent quintic of the modular equation of the 79^{th} order.)

§3. In the preceding sections it has been assumed that the quintic is of the form

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0.$$

If we consider the full-termed form

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0,$$

the course of the solution will follow that in the preceding investigation, step by step, and may be summarized as follows:

Write

$$
\begin{aligned}
H_2 &= ac - b^2, \\
H_3 &= a^2d - 3abc + 2b^3, \\
H_4 &= a^2(ae - 4bd + 3c^2), \\
H_5 &= a^2(a^2f - 5abe + 2acd + 8b^2d - 6bc^2), \\
H_6 &= H_2H_4 - H_3^2 - 4H_2^3, \\
H_7 &= H_2H_5 - H_3H_4, \\
H_8 &= H_3H_5 + 6H_2H_6 - H_4^2 + 4H_2^2H_4, \\
H_9 &= 2H_2H_7 - 3H_3H_6, \\
H_{10} &= H_5^2 + 12H_4H_6 + 4H_2H_4^2, \\
H_{12} &= H_2H_{10} - 2H_4H_8 + 9H_6^2 - H_4^3, \\
H_{13} &= H_5H_8 + 2H_4H_9 + 24H_2^5H_3, \\
H_{20} &= 2H_9(H_4H_7 - 3H_5H_6) + H_8(2H_4H_8 - H_5H_7 - 15H_6^2) - 3H_4^2(H_6^2 + 8H_2^6), \\
ax &= -b + \omega y_1 + \omega^2 y_2 + \omega^3 y_3 + \omega^4 y_4, \\
&\qquad \omega^5 - 1 = 0,
\end{aligned}
$$

and as before,

$$
\begin{aligned}
2\sigma_2 &= y_1y_4 + y_2y_3, \\
2\sigma_3 &= y_1^2y_3 + y_2^2y_1 + y_3^2y_4 + y_4^2y_2, \\
2\sigma_4 &= y_1^3y_2 + y_2^3y_4 + y_3^3y_1 + y_4^3y_3, \\
2\sigma_5 &= y_1^5 + y_2^5 + y_3^5 + y_4^5, \\
2\tau_2 &= y_1y_4 - y_2y_3, \\
2\tau_3 &= y_1^2y_3 - y_2^2y_1 - y_3^2y_4 + y_4^2y_2, \\
u &= 25\tau_2^2,
\end{aligned}
$$

then will

$$
\begin{aligned}
\sigma_2 + H_2 &= 0, \\
\sigma_3 + H_3 &= 0, \\
2\sigma_4 - \sigma_2^2 - 3\tau_2^2 + H_4 - 3H_2^2 &= 0, \\
2\sigma_5 - 20\tau_2\tau_3 + H_5 - 2H_2H_3 &= 0.
\end{aligned}
$$

Eliminating σ_2, σ_3, σ_4 and σ_5, we obtain

$$
\tau_3 = \frac{\{H_5u^2 - 2(4H_9 + H_4H_5)u - H_{13} + 50H_6H_7\}\tau_2}{u_3 - 3H_4u^2 - (2H_8 - H_4^2)u - H_{12} + 25H_6^2}
$$

and

$$
\begin{aligned}
&[\{u^3 - 3H_4u^2 - (2H_8 - H_4^2)u - H_{12} + 25H_6^2\} \\
&\quad \times \{H_2u^4 - (H_6 + 7H_2H_4)u^3 + (H_{10} - 18H_2H_8 - 5H_4H_6 + 9H_2H_4^2)u^2 \\
&\qquad - (H_4H_{10} + 21H_7^2 - 66H_6H_8 - 26H_4^2H_6 + 464H_2H_6^2)u + 25H_6(H_{12} - 25H_6^2)\} \\
&\quad - \{H_5u^2 - 2(4H_9 + H_4H_5)u - H_{13} + 50H_6H_7\}^2 u] \div (H_2u - H_6) = 0,
\end{aligned}
$$

which can be reduced to the form

$$\{u^3 - 5H_4u^2 - 5(2H_8 - H_4^2)u - 5(H_{12} - 25H_6^2)\}^2 - (128H_{20} + H_{10}^2)u = 0. \quad (4)$$

Hence, if

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}[-(H_3 - \tau_3) + \sqrt{\{(H_3 - \tau_3)^2 + 4(H_2 - \tau_2)(H_2^2 - \tau_2^2)\}}] \\ \alpha_2 &= \qquad\qquad \alpha_3 = \qquad\qquad \alpha_4 = \end{aligned}$$

then will

$$ax = -b + \omega\left\{\frac{\alpha_1^2\alpha_3}{(H_2 - \tau_2)^2}\right\}^{\frac{1}{5}} + \text{etc.}$$

THEOREM I.—Every algebraic equation A of degree n has a dioristic equation B of degree $(n-2)!$ whose coefficients are seminvariants of the coefficients of A, and which is such that if A be solvable by radicals, B will have a rational root, and conversely, if B have a rational root, A will be solvable by radicals.

THEOREM II.—If the equation A be solvable by radicals, its coresolvent equations will be of the degrees whose indices are the prime factors of $n-1$.

Examples.—The dioristic equation of the cubic

$$ax^3 + 3bx^2 + 3cx + d = 0$$

is

$$z + H_2 = 0,$$

and the coresolvent equation is a quadratic.

The dioristic equation of the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$$

is

$$4z^2 - H_3^2 = 0,$$

and the coresolvent equation is a cubic.

The dioristic equation of the quintic is reducible by a linear transformation to equation (4) above, and the coresolvent equations are two quadratics.

OTTAWA, CANADA, *March*, 1900.

Construction of the Geometry of Euclidean n-Dimensional Space by the Theory of Continuous Groups.

By E. O. Lovett.

1. With regard to space, let it be assumed:

1°. That it is an n-dimensional manifoldness, i. e., that n independent data are necessary and sufficient to determine the position of an element of the manifoldness; these n independent things are called the coordinates of the element.

2°. That a figure of the manifoldness possesses $\frac{n(n+1)}{2}$ degrees of freedom within the manifoldness; i. e., that $\frac{n(n+1)}{2}$ independent data are necessary and sufficient to render a rigid body fixed in position; the latter $\frac{n(n+1)}{2}$ independent things are called the parameters of the figure.

For convenience, let the element be called a point, and its coordinates be designated by $x_1, x_2, \ldots, x_n$. Consider any figure containing this point and let the parameters of the figure be $a_1, a_2, \ldots, a_{\frac{n(n+1)}{2}}$. Let the figure assume a new position and call $x_1', x_2', \ldots, x_n'$ the coordinates of the new position of $(x_1, x_2, \ldots, x_n)$. Then

$$x_i' = \xi_i(x_1, x_2, \ldots, x_n, \quad a_1, a_2, \ldots, a_{\frac{n(n+1)}{2}}),$$
$$i = 1, 2, \ldots, n.$$

The operation changing $(x_1, x_2, \ldots, x_n)$ into $(\xi_1, \xi_2, \ldots, \xi_n)$ represents one of the motions of an n-dimensional figure, and the ensemble of all these operations constitutes a continuous group with $\frac{n(n+1)}{2}$ parameters. The iden-

tical transformation ought to appear among these operations; accordingly, there must be some system of parameters $a_1, \ldots, a_{\frac{n(n+1)}{2}}$, such that

$$\xi_1 = x_1, \quad \xi_2 = x_2, \ldots, \xi_n = x_n.$$

There is no loss of generality in assuming the preceding system of values to be

$$a_1 = a_2 = \ldots = a_{\frac{n(n+1)}{2}} = 0.$$

An infinitesimal transformation of the group is one whose parameters differ by infinitesimals from those parameters which produce the identical transformation; in this case, the infinitesimal transformation is obtained by assigning infinitesimal values to the parameters themselves; i. e., by such a transformation $x_1, x_2, \ldots, x_n$ are changed respectively to

$$x_i + \sum_1^{\frac{n(n+1)}{2}} {}_j\, a_j \frac{\partial \xi_i}{\partial a_j}, \qquad i = 1, 2, \ldots, n,$$

in the partial derivatives of which the a's should be put equal to zero.

Writing, for short,

$$p_i = \frac{\partial f}{\partial x_i}, \qquad i = 1, 2, \ldots, n,$$

the common symbol for the above infinitesimal transformation is

$$I \equiv \sum_1^n {}_i\, p_i \sum_1^{\frac{n(n+1)}{2}} {}_j\, a_j \frac{\partial \xi_i}{\partial a_j},$$

or, putting

$$J_i \equiv \sum_1^n {}_j\, p_j \frac{\partial \xi_j}{\partial a_i}, \qquad i = 1, 2, \ldots, \frac{n(n+1)}{2},$$

any infinitesimal transformation may be written

$$I \equiv \sum_1^{\frac{n(n+1)}{2}} a_i J_i.$$

By a fundamental theorem of Lie, if we put

$$(J_i, J_j) \equiv \sum_1^n {}_k \left(\frac{\partial J_i}{\partial p_k} \frac{\partial J_j}{\partial x_k} - \frac{\partial J_i}{\partial x_k} \frac{\partial J_j}{\partial p_k} \right),$$

then

$$(J_i, J_j) = \sum_{1}^{\frac{n(n+1)}{2}} {}^{k} \lambda_k^{(i,j)} J_k , \tag{1}$$

$$i = 1, 2, \ldots, \frac{n(n+1)}{2}, \quad j = 1, 2, \ldots, \frac{n(n+1)}{2},$$

where the λ's are constants. There are

$$\tfrac{1}{8} \{n^2(n+1)^2 - 2n(n+1)\}$$

of these equations, but the

$$\tfrac{1}{16} \{n^3(n+1)^3 - 2n^2(n+1)^2\}$$

λ's are not wholly arbitrary, since the following

$$\tfrac{1}{48} \{n^3(n+1)^3 - 6n^2(n+1)^2 + 8n(n+1)\}$$

identities of Jacobi

$$(J_i, (J_j, J_k)) + (J_j, (J_k, J_i)) + (J_k, (J_i, J_j)) = 0, \tag{2}$$

$$i, j, k = 1, 2, \ldots, \frac{n(n+1)}{2}$$

must hold.

Every set of J's satisfying (1) and (2) reveals a space whose independent infinitesimal motions are represented by the infinitesimal operators of the set. Those functions of the elements which are invariant under these transformations will characterize the geometry of the space. It is proposed here to find these characteristics for one set of operators.

2. The following forms for the fundamental transformations

$$J_1, J_2, \ldots J_{\frac{n(n+1)}{2}}$$

$$p_1, p_2, \ldots, p_n, \quad x_i p_j - x_j p_i, \tag{3}$$

$$i, j = C_{n,2} \text{ of } 1, 2, \ldots, n,$$

satisfy the conditions (1) and (2).

Let $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ be any two points, and $(\xi_1, \xi_2, \ldots, \xi_n)$, $(\xi_1', \xi_2', \ldots, \xi_n')$ be their positions after they are subjected to the transformation

$$I \equiv \sum_{1}^{\frac{n(n+1)}{2}} J_i,$$

where the J_i's have the values (3).

Let $\phi(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ be a function which is absolutely invariant under this operation; then, if such a function exist, we must have

$$\phi(\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n) = \phi(x_1, \ldots, x_n, x'_1, \ldots, x'_n),$$

that is,

$$I\phi \equiv 0$$

for all values of the α's; hence, equating to zero the coefficients of the α's in $I\phi$, we have the following system of linear partial differential equations for the function ϕ:

$$\left.\begin{array}{ll} \dfrac{\partial\phi}{\partial x_i} + \dfrac{\partial\phi}{\partial x'_i} = 0, & i = 1, 2, \ldots, n, \\ x_i \dfrac{\partial\phi}{\partial x_j} + x'_i \dfrac{\partial\phi}{\partial x'_j} - x'_j \dfrac{\partial\phi}{\partial x_i} - x'_j \dfrac{\partial\phi}{\partial x'_i} = 0, & \\ \qquad i, j = C_{n,2} \text{ of } 1, 2, \ldots, n. & \end{array}\right\} \quad (4)$$

This system of $\dfrac{n(n+1)}{2}$ equations in $2n$ variables should not, in general, possess a solution. But the equations are not all independent. In fact, if we take the following system of $2n-1$ equations from the above, namely,

$$\frac{\partial\phi}{\partial x_\rho} + \frac{\partial\phi}{\partial x'_\rho} = 0, \qquad \rho = 1, 2, \ldots, n, \quad (5)$$

$$x_1 \frac{\partial\phi}{\partial x_\sigma} - x_\sigma \frac{\partial\phi}{\partial x_1} + x'_1 \frac{\partial\phi}{\partial x'_\sigma} - x'_\sigma \frac{\partial\phi}{\partial x'_1} = 0, \qquad \sigma = 2, 3, \ldots, n; \quad (6)$$

multiply the equations (5) respectively by

$$\frac{l_h^{(i,j)}}{x_1 - x'_1}, \qquad h = 1, 2, \ldots, n,$$

where

$$l_1^{(i,j)} = x_i x'_j - x_j x'_i, \quad l_k = 0, \quad k = 2, \ldots, n; \ k \neq i \neq j$$
$$l_i^{(i,j)} = x_j x'_1 - x_1 x'_j, \quad l_j^{(i,j)} = x_1 x'_i - x_i x'_1;$$

multiply the equations (6) respectively by

$$\frac{\lambda_h^{(i,j)}}{x_1 - x'_1}, \qquad h = 2, 3, \ldots, n$$

where

$$\lambda_i^{(i,j)} = x'_j - x_j, \quad \lambda_j^{(i,j)} = x_i - x'_i,$$
$$\lambda_k^{(i,j)} = 0, \quad k = 2, 3, \ldots, n; \ k \neq i \neq j;$$

and add the results, we obtain the remaining $\frac{1}{2}(n-1)(n-2)$ equations of the system (4), namely,

$$x_i \frac{\partial \phi}{\partial x_j} - x_j \frac{\partial \phi}{\partial x_i} + x_i' \frac{\partial \phi}{\partial x_j'} - x_j' \frac{\partial \phi}{\partial x_i'} = 0 .$$

The complete system (5) and (6) of $2n-1$ equations in $2n$ variables possesses at least one solution. That it has no more is readily seen by noting the fact that not all $(2n-1)^{\text{th}}$ order determinants of the matrix of $2n$ columns

$$\left\|\begin{array}{cccccccccc}
0 & 0 & 0 & 0 \ldots\ldots & 0 & 1 & 0 & 0 & 0 \ldots\ldots & 0 \\
0 & 0 & 0 & 0 \ldots\ldots & 0 & 0 & 1 & 0 & 0 \ldots\ldots & 0 \\
\multicolumn{10}{c}{\dotfill} \\
0 & 0 & 0 & 0 \ldots\ldots & 0 & 0 & 0 & 0 & 0 \ldots\ldots & 1 \\
x_2'-x_2 & x_1'-x_1 & 0 & 0 \ldots\ldots & 0 & -x_2' & x_1' & 0 & 0 \ldots\ldots & 0 \\
x_3'-x_3 & 0 & x_1'-x_1 & 0 \ldots\ldots & 0 & -x_3' & 0 & x_1' & 0 \ldots\ldots & 0 \\
\multicolumn{10}{c}{\dotfill} \\
x_n'-x_n & 0 & 0 & 0 \ldots\ldots & x_1'-x_1 & -x_n' & 0 & 0 & 0 \ldots\ldots & x_1'
\end{array}\right\|$$

vanish; for example, the one formed by the last $2n-1$ columns whose value is $(x_1-x_1')^{n-1}$.

The unique solution of this system appears by observing that the first n equations demand that the solution be a function of

$$X_i = x_i - x_i', \qquad i = 1, 2, \ldots, n.$$

In these new variables the last $n-1$ equations become

$$X_i \frac{\partial \phi}{\partial X_j} - X_j \frac{\partial \phi}{\partial X_i} = 0, \qquad i, j, = 1, 2, \ldots, n,$$

which require that ϕ be a function of

$$\sum_1^n X_i^2;$$

that is, the function

$$\delta = \sqrt{\sum_1^n (x_i - x_i')^2} \tag{7}$$

is an absolute invariant under the most general tranformation of the group (3).

This function δ is said to define the distance between the two points $(x_1, x_2, \ldots, x_n)$, $(x_1', x_2', \ldots, x_n')$.

3. Consider, now, the linear manifoldness of elements

$$x_i + \lambda_i x_1 + \alpha_i = 0, \qquad i = 2, 3, \ldots, n.$$

The increments assigned to $x_1, x_2, \ldots, x_n$ by the transformations (3) are as follows:

$$\left.\begin{array}{ll} \text{By } p_j & \delta x_i = 1,\ i = j,\ \delta x_i = 0, i \neq j, \\ & i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n; \\ \text{By } x_i p_j - x_j p_i & \delta x_i = -x_j, \quad \delta x_j = +x_i, \quad \delta x_k = 0, \\ & k \neq i \neq j = 1, 2, \ldots, n. \end{array}\right\} \tag{9}$$

By forming the variational equations

$$\delta(x_i + \lambda_i x_1 + \alpha_i) = 0$$

and substituting therein the values (9), we find the following values for the variations of λ_i and α_i under the transformations (3):

$$\left.\begin{array}{ll} \text{Under } p_i & \left\{\begin{array}{l} i = 1, \qquad \delta\lambda_j = 0, \quad \delta\alpha_j = -\lambda_j, \\ \qquad j = 2, 3, \ldots, n; \\ i = 2, 3, \ldots, n, \quad \delta\lambda_j = 0, \quad \delta\alpha_i = -1, \quad \delta\alpha_j = 0, \\ \qquad j = 2, 3, \ldots, n; \end{array}\right. \\ \text{Under } x_i p_j - x_j p_i & \left\{\begin{array}{l} i \neq 1, j \neq 1 \left\{\begin{array}{l} \delta\lambda_i = -\lambda_j, \quad \delta\lambda_j = \lambda_i, \quad \delta\alpha_i = -\alpha_j, \\ \qquad \delta\alpha_j = \alpha_i, \\ \delta\lambda_k = \delta\alpha_k = 0, \ k = 2, 3, \ldots, n \neq i \neq j; \end{array}\right. \\ i = 2, 3, \ldots, n,\ j = 1,\ \delta\lambda_i = \lambda_i^2 + 1,\ \delta\lambda_k = \lambda_i\lambda_k, \\ k = 2, 3, \ldots, n \neq i,\ \delta\alpha_{k'} = \alpha_i\lambda_{k'},\ k' = 2, 3, \ldots, n. \end{array}\right. \end{array}\right\} \tag{10}$$

The invariants of two linear manifoldnesses

$$(\lambda_2, \lambda_3, \ldots \lambda_n, \alpha_2, \alpha_3, \ldots, \alpha_n), \quad (\lambda_2', \lambda_3', \ldots \lambda_n', \alpha_2', \alpha_3', \ldots, \alpha_n')$$

are solutions of the system of partial differential equations

$$\sum_n^2{}_i \left\{ \frac{\partial\phi}{\partial\lambda_i} \delta_j\lambda_i + \frac{\partial\phi}{\partial\alpha_i} \delta_j\alpha_i + \frac{\partial\phi}{\partial\lambda_i'} \delta_j\lambda_i' + \frac{\partial\phi}{\partial\alpha_i'} \delta_j\alpha_i \right\} = 0, \tag{11}$$

$$j = 1, 2, \ldots, \frac{n(n+1)}{2},$$

where the increments δ_j are to be successively replaced by those due to the $\frac{n(n+1)}{2}$ infinitesimal transformations (3) as tabulated above in (10).

For convenience, the equations (11) are arranged in the following manner: $(n-1)$ equations:

$$\frac{\partial\phi}{\partial\alpha_i}+\frac{\partial\phi}{\partial\alpha'_i}=0, \qquad i=2,3,\ldots,n; \tag{12}$$

1 equation:

$$\sum_2^n\left(\lambda_i\frac{\partial\phi}{\partial\alpha_i}+\lambda'_i\frac{\partial\phi}{\partial\alpha'_i}\right)=0; \tag{13}$$

$(n-2)$ equations:

$$\lambda_2\frac{\partial\phi}{\partial\lambda_j}-\lambda_j\frac{\partial\phi}{\partial\lambda_2}+\alpha_2\frac{\partial\phi}{\partial\alpha_j}-\alpha_j\frac{\partial\phi}{\partial\alpha_2}+\lambda'_2\frac{\partial\phi}{\partial\lambda'_j}-\lambda'_j\frac{\partial\phi}{\partial\lambda'_2}$$
$$+\alpha'_2\frac{\partial\phi}{\partial\alpha'_j}-\alpha'_j\frac{\partial\phi}{\partial\alpha'_2}=0, \qquad j=3,4,\ldots,n; \tag{14}$$

$\frac{1}{2}(n-2)(n-3)$ equations:

$$\lambda_i\frac{\partial\phi}{\partial\lambda_j}-\lambda_j\frac{\partial\phi}{\partial\lambda_i}+\alpha_i\frac{\partial\phi}{\partial\alpha_j}-\alpha_j\frac{\partial\phi}{\partial\alpha_i}+\lambda'_i\frac{\partial\phi}{\partial\lambda'_j}-\lambda'_j\frac{\partial\phi}{\partial\lambda'_i}$$
$$+\alpha'_i\frac{\partial\phi}{\partial\alpha'_j}-\alpha'_j\frac{\partial\phi}{\partial\alpha'_i}=0, \qquad i=3,4,\ldots,n,\ j=2,3,\ldots,n; \tag{15}$$

$(n-1)$ equations:

$$(\lambda_j^2+1)\frac{\partial\phi}{\partial\lambda_j}+(\lambda_j'^2+1)\frac{\partial\phi}{\partial\lambda'_j}+\sum_{i=2\neq j}^{i=n\neq j}\left(\lambda_j\lambda_i\frac{\partial\phi}{\partial\lambda_i}+\lambda'_j\lambda'_i\frac{\partial\phi}{\partial\lambda'_i}\right)$$
$$+\sum_{i=2}^{i=n}\alpha_j\alpha_i\frac{\partial\phi}{\partial\alpha_i}+\alpha'_j\alpha'_i\frac{\partial\phi}{\partial\alpha'_i}\Big)=0 \qquad j=2,3,\ldots,n. \tag{16}$$

It will be seen from table (10) that the variations of the λ's are wholly independent of the α's; accordingly, invariant functions of the λ's and λ''s alone may exist. If such invariant functions $\psi(\lambda_2,\ldots,\lambda_n,\lambda'_2,\ldots,\lambda'_n)$ do exist they are solutions of the system of $\frac{n(n-1)}{2}$ equations in $2(n-1)$ variables:

$\frac{(n-2)(n-3)}{2}$ equations:

$$\lambda_i\frac{\partial\psi}{\partial\lambda_j}-\lambda_j\frac{\partial\psi}{\partial\lambda_i}+\lambda'_i\frac{\partial\psi}{\partial\lambda'_j}-\lambda'_j\frac{\partial\psi}{\partial\lambda'_i}=0, \tag{17}$$
$$i=3,4,\ldots,n,\ j=2,3,\ldots,n;$$

$(n-2)$ equations:

$$\lambda_2 \frac{\partial \psi}{\partial \lambda_j} - \lambda_j \frac{\partial \psi}{\partial \lambda_2} + \lambda_2' \frac{\partial \psi}{\partial \lambda_j'} - \lambda_j' \frac{\partial \psi}{\partial \lambda_2'} = 0, \qquad (18)$$

$$j = 3, 4, \ldots, n;$$

$(n-1)$ equations:

$$(\lambda_j^2 + 1) \frac{\partial \psi}{\partial \lambda_j} + (\lambda_j'^2 + 1) \frac{\partial \psi}{\partial \lambda_j'} + \sum_{i=2 \neq j}^{i=n \neq j} \left(\lambda_j \lambda_i \frac{\partial \psi}{\partial \lambda_i} + \lambda_j' \lambda_i' \frac{\partial \psi}{\partial \lambda_i'} \right) = 0, \qquad (19)$$

$$j = 2, 3, , \ldots, n.$$

This system cannot possess a solution if the equations are all independent. That the number of independent equations in the system is not greater than $2n-3$ we verify by observing that the $\frac{(n-2)(n-3)}{2}$ determinants of the $2(n-1)^{\text{th}}$ order formed by appending the rows of the array:

$$\left\{\begin{matrix}
0 & -\lambda_n & 0 & 0 \cdots & 0 & 0 & \lambda_3 & 0 & -\lambda_n' & 0 & 0 \cdots & 0 & 0 & \lambda_3' \\
0 & 0 & -\lambda_n & 0 \cdots & 0 & 0 & \lambda_4 & 0 & 0 & -\lambda_n' & 0 \cdots & 0 & 0 & \lambda_4' \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 \cdots & 0 & -\lambda_n & \lambda_{n-1} & 0 & 0 & 0 & 0 \cdots & 0 & -\lambda_n' & \lambda_{n-1}' \\
0 & -\lambda_{n-1} & 0 & 0 \cdots & 0 & \lambda_3 & 0 & 0 & -\lambda_{n-1}' & 0 & 0 \cdots & 0 & \lambda_3' & 0 \\
0 & 0 & -\lambda_{n-1} & 0 \cdots & 0 & \lambda_4 & 0 & 0 & 0 & -\lambda_{n-1}' & 0 \cdots & 0 & \lambda_4' & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 \cdots & -\lambda_{n-1} & \lambda_{n-2} & 0 & 0 & 0 & 0 & 0 \cdots & -\lambda_{n-1}' & \lambda_{n-2}' & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\lambda_4 & \lambda_3 & 0 \cdots & 0 & 0 & 0 & 0 & -\lambda_4' & \lambda_3' & 0 \cdots & 0 & 0 & 0
\end{matrix}\right\} \qquad (20)$$

successively to the matrix

$$\left\Vert\begin{matrix}
-\lambda_3 & \lambda_2 & 0 & 0 & \cdots & 0 & -\lambda_3' & \lambda_2' & 0 & 0 & \cdots & 0 \\
-\lambda_4 & 0 & \lambda_2 & 0 & \cdots & 0 & -\lambda_4' & 0 & \lambda_2' & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\lambda_n & 0 & 0 & 0 & \cdots & \lambda_2 & -\lambda_n' & 0 & 0 & 0 & \cdots & \lambda_2' \\
\lambda_2^2+1 & \lambda_2\lambda_3 & \lambda_2\lambda_4 & \lambda_2\lambda_5 & \cdots & \lambda_2\lambda_n & \lambda_2'^2+1 & \lambda_2'\lambda_3' & \lambda_2'\lambda_4' & \lambda_2'\lambda_5' & \cdots & \lambda_2'\lambda_n' \\
\lambda_2\lambda_3 & \lambda_3^2+1 & \lambda_3\lambda_4 & \lambda_3\lambda_5 & \cdots & \lambda_3\lambda_n & \lambda_2'\lambda_3' & \lambda_3'^2+1 & \lambda_3'\lambda_4' & \lambda_3'\lambda_5' & \cdots & \lambda_3'\lambda_n' \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_2\lambda_n & \lambda_3\lambda_n & \lambda_4\lambda_n & \lambda_5\lambda_n & \cdots & \lambda_n^2+1 & \lambda_2'\lambda_n' & \lambda_3'\lambda_n' & \lambda_4'\lambda_n' & \lambda_5'\lambda_n' & \cdots & \lambda_n'^2+1
\end{matrix}\right\Vert \qquad (21)$$

are equal to zero.

Consider, then, the system formed by the $(2n-3)$ equations (18) and (19) in the $2(n-1)$ variables $\lambda_2, \ldots, \lambda_n, \lambda_2', \ldots, \lambda$.

The equations (18) assert that ψ is a function of

$$\xi \equiv \sum_2^n \lambda_i^2, \quad \eta \equiv \sum_2^n \lambda_i'^2, \quad \zeta \equiv \sum_2^n \lambda_i \lambda_i'. \tag{22}$$

The equations (19) become in the new variables ξ, η, ζ:

$$2\lambda_j \xi_1 \frac{\partial \phi}{\partial \xi_1} + 2\lambda_j' \eta_1 \frac{\partial \phi}{\partial \eta_1} + (\lambda_j + \lambda_j') \zeta_1 \frac{\partial \phi}{\partial \zeta_1} = 0. \tag{23}$$

$$j = 2, 3, \ldots, n,$$

where $\qquad \xi_1 = \xi + 1, \quad \eta_1 = \eta + 1, \quad \zeta_1 = \zeta + 1.$

From any one of these last $(n-1)$ equations we have

$$\frac{d\xi_1}{\xi_1} + \frac{d\eta_1}{\eta_1} - 2\frac{d\zeta_1}{\zeta_1} = 0,$$

that is, ϕ is a function of

$$\zeta_1^2 / \xi_1 \eta_1,$$

or the quantity

$$\cos^2\theta = \frac{\left\{\sum_2^n (\lambda_i \lambda_i' + 1)\right\}^2}{\left\{\sum_2^n \lambda_i^2 + 1\right\}\left\{\sum_2^n \lambda_i'^2 + 1\right\}} \tag{24}$$

is an absolute invariant under the most general transformation of the group (3). The angle θ is said to be the angle between the two linear manifoldnesses.

That the above solution is unique is seen from the fact that not every $2(n-1)^{\text{th}}$ order determinant of the matrix (21) vanishes.

As appears in the table (10), the variations of the α's are functions of the α's and λ's; we should not expect, then, to find an invariant function of the α's alone.

But the system composed of the equations (12), (13), (14), (15) and (16) yields $(n-1)$ invariants, functions of the α's and λ's, which come to light in the following manner:

The equations (12) assert that ϕ is a function of

$$\rho_i \equiv \alpha_i - \alpha_i', \qquad i = 2, 3, \ldots, n.$$

In the latter variables the equation (13) becomes

$$\sum_2^n \sigma_j \frac{\partial \phi}{\partial \rho_j} = 0, \quad \sigma_j = \lambda_j - \lambda_j'. \qquad j = 2, 3, \ldots, n,$$

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which requires that ϕ be a function of any set of the $(n-1)$ sets of $(n-2)$ determinants each,

$$\begin{aligned}
&\rho_2\sigma_j - \rho_j\sigma_2 \equiv (\rho\sigma)_{2j}, && j = 2, 3, \ldots, n;\\
&\rho_3\sigma_j - \rho_j\sigma_3 \equiv (\rho\sigma)_{3j}, && j = 2, 3, \ldots, n;\\
&\ldots\ldots\ldots\ldots\ldots\ldots, &&\\
&\rho_n\sigma_j - \rho_j\sigma_n \equiv (\rho\sigma)_{nj}, && j = 2, 3, \ldots, n.
\end{aligned}$$

The $\frac{(n-1)(n-2)}{2}$ equations (14) and (15) become the $(n-1)$ following:

$$(\rho\sigma)_{ki}\frac{\partial\phi}{\partial(\rho\sigma)_{kj}} - (\rho\sigma)_{kj}\frac{\partial\phi}{\partial(\rho\sigma)_{ki}} = 0, \qquad ij = 2, 3, \ldots, n,$$

which demand that ϕ be a function of

$$P \equiv \sum_{i=2}^{i=n}(\rho\sigma)^2_{ki};$$

furthermore, in the original variables, the equations (14) and (15) assert that the λ's and λ''s enter into ϕ alone only by means of the determinants

$$\left\|\begin{matrix} \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_n & 1\\ \lambda'_2 & \lambda'_3 & \lambda'_4 & \ldots & \lambda'_n & 1\end{matrix}\right\|$$

and, in fact, only through the forms

$$Q \equiv \left\|\begin{matrix} \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_n\\ \lambda'_2 & \lambda'_3 & \lambda'_4 & \ldots & \lambda_n\end{matrix}\right\|^2,$$

and

$$R \equiv \sum_2^n(\lambda_i - \lambda'_i)^2;$$

further, the equations (16) cannot be satisfied unless R and Q enter into the combination

$$Q + R \equiv \left\|\begin{matrix} \lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_n & 1\\ \lambda'_2 & \lambda'_3 & \lambda'_4 & \ldots & \lambda'_n & 1\end{matrix}\right\|^2 \equiv T.$$

Finally, in the variables P and T the last $(n-1)$ equations assume the form

$$(\lambda_j + \lambda'_j)\left\{P\frac{\partial\phi}{\partial P} + T\frac{\partial\phi}{\partial T}\right\} = 0, \qquad j = 2, 3, \ldots, n,$$

that is, the solution ϕ is an arbitrary function of

$$P/T.$$

Accordingly, the following $(n-1)$ expressions

$$\Delta^2 \equiv \frac{\sum\limits_{k=2}^{k=n} \begin{vmatrix} \lambda_j - \lambda'_j & \lambda_k - \lambda'_k \\ a_j - a'_j & a_k - a'_k \end{vmatrix}^2}{\begin{Vmatrix} \lambda_2 & \lambda_3 & \lambda_4 & \dots & \lambda_n & 1 \\ \lambda'_2 & \lambda'_3 & \lambda'_4 & \dots & \lambda'_n & 1 \end{Vmatrix}^2}, \qquad j = 2, 3, \dots n,$$

are invariant functions of the parameters of the two linear manifoldnesses under the most general transformation of the group (3).

When the two linear manifoldnesses intersect, all the Δ_j's are zero; conversely, when any Δ_j is zero, all the others are zero, and the linear manifoldnesses have a point in common.

It is clear, *a fortiori*, that the form

$$\Delta^2 \equiv \frac{\begin{Vmatrix} \lambda_2 - \lambda'_2 & \lambda_3 - \lambda'_3 & \dots & \lambda'_n - \lambda'_n \\ a_2 - a'_2 & a_3 - a'_3 & \dots & a_n - a'_n \end{Vmatrix}^2}{\begin{Vmatrix} \lambda_2 & \lambda_3 & \lambda_4 & \dots & \lambda_n & 1 \\ \lambda'_2 & \lambda'_3 & \lambda'_4 & \dots & \lambda'_n & 1 \end{Vmatrix}^2}$$

is an invariant. The analogue of the forms Δ_j and Δ to the expression for the distance between the straight lines in ordinary space is apparent.

The fundamental notions of distance and direction in space of n-dimensions are thus introduced by the invariants (7) and (24), the former relative to two elements of the space, the latter relative to two simplest manifoldnesses composed of a simply infinite number of these elements. All the derived notions of geometry may then be derived by as simple extensions of these primary notions successively from three dimensions as occur when passing from the plane to ordinary space.

PRINCETON, NEW JERSEY.

A Table of Class Numbers for Cubic Number Fields.

By Legh W. Reid.

This table has been calculated with a view to furnishing for the general algebraic number fields an amount of number material sufficiently great to be of use in the further study of these fields, and in particular in that of the cubic fields. It gives for each of 161 cubic number fields the class number, h, the discriminant Δ, a basis, and the factorization of certain rational primes into their prime ideal factors. When $h = 1$, the prime *number* factors of these primes are given. Units are also given for most of the fields. The method employed in the calculation of the class numbers is to be looked at from the point of view of the practicability of carrying out the numerical reckoning involved, and the actual determination of the numerical value of h. It is to be sharply distinguished from those representations of h by an infinite series, which, although theoretically perfect, lead in only a very limited number of cases to the determination of the numerical value of h.

The method used depends upon the following theorem of Minkowski's:

Th. 1.—*In every ideal class there is an ideal,* $\mathfrak{i}$, *whose norm,* $n(\mathfrak{i})$, *satisfies the condition*

$$n(\mathfrak{i}) < \left(\frac{4}{\pi}\right)^r \frac{m!}{m^m} \left|\sqrt{\Delta}\right|, \tag{1}$$

where m is the degree and Δ *the discriminant of the field, and r the number of pairs of imaginary fields found among the m conjugate fields* $k^{(1)}, k^{(2)}, \ldots, k^{(m)}$. I shall denote $\left(\frac{4}{\pi}\right)^r \frac{m!}{m^m}$ by M. If then we find all ideals of a proposed field, k, whose norms satisfy the condition (1), and determine the equivalences which exist between them; i. e., into how many ideal classes they fall, we have determined the class number of k.

The task may be divided into two parts:

I. The obtaining of all ideals whose norms satisfy (1).

II. The determination of the number of the ideal classes into which they fall.

Let θ be an integer defining the body k, $f(x)=0$, the equation of lowest degree, the m^{th}, with rational coefficients, satisfied by θ, and $d(\theta)$ the discriminant of this equation.

We must first of all determine a basis and the discriminant Δ of k.

It can be easily shown that if $d(\theta)$ be not divisible by the square of a rational integer, then

$$d(\theta)=\Delta$$

and $1, \theta^2, \ldots, \theta^{m-1}$ is a basis of k.

In the case of *cubic bodies*, when $d(\theta)$ is divisible by the square of a rational integer, we may determine a basis and hence Δ by a method given by Woronoj.

I. We obtain all ideals of k, whose norms satisfy (1) in the following manner: Since the norm of a prime ideal is a power of the rational prime, which it divides, we shall obtain all prime ideals, whose norms satisfy (1), if we factor into their prime ideal factors all rational primes $< M|\sqrt{\Delta}|$.

The desired ideals are then such of these prime ideals, their powers and products as satisfy (1). We have, then, first of all to factor all rational primes $< M|\sqrt{\Delta}|$.

This is easily accomplished in the case $\dfrac{d(\theta)}{\Delta} \not\equiv 0 \pmod{p}$ by means of the following theorem:

TH. 2.—*If p satisfy the condition $\dfrac{d(\theta)}{\Delta} \not\equiv 0 \pmod{p}$, and if we resolve the left-hand member of $f(x)=0$ into its prime factors with respect to the modulus p, as*

$$f(x) \equiv \{P(x)\}^e\{P'(x)\}^{e'}, \ldots, \pmod{p},$$

where $P(x)$, $P'(x)$, ... are different prime functions with respect to p, and of degrees f, f', ... respectively, then is

$$(p)=(p, P(\theta))^e\,(p, P'(\theta))^{e'} \ldots$$

the required factorization of (p), where $(p, P(\theta))$, $(p, P'(\theta))$ are different prime ideals of degrees f, f', ... respectively.

When $\dfrac{d(\theta)}{\Delta} \equiv 0 \pmod{p}$, the factorization of (p) may be effected in the case of *cubic bodies* by a method given by Woronoj.

I shall consider, during the remainder of this paper, the body under discussion to be cubic.

II. We now take up the determination of the number of ideal classes into which fall the ideals of k whose norms satisfy (1).

The method that I have used is the following:

Having selected any prime ideal $\mathfrak{p}$ whose norm satisfies the above condition, we must determine first of all whether $\mathfrak{p}$ is a principal ideal. The following method answers not only this question but the more general one. What is the lowest power of $\mathfrak{p}$, which is a principal ideal?

We find an integer α such that $(\alpha) = \mathfrak{p}^n$. $\mathfrak{p}$ is evidently a principal or non-principal ideal according as (α) is or is not the n^{th} power of a principal ideal. If $n = 1$, $\mathfrak{p} = (\alpha)$, a principal ideal.

The necessary and sufficient condition that (α) be the n^{th} power of a principal ideal is that a unit η exist such that $\alpha\eta$ is the n^{th} power of an integer.

That is (α) can be the n^{th} power of a principal ideal, although α is not the n^{th} power of an integer, but acquires this property through multiplication by a suitable unit.

The following theorems simplify the determination of the lowest power of $\mathfrak{p}$, which is a principal ideal:

TH. 3.—*If the m_1^{th} and m_2^{th} powers of an ideal $\mathfrak{a}$ be principal ideals, then is also the l^{th} power of $\mathfrak{a}$ a principal ideal, where l is the greatest common divisor of m_1 and m_2.*

TH. 4.—*The necessary and sufficient condition, that $\mathfrak{a}^n = (\alpha)$ be the lowest power of $\mathfrak{a}$, which is a principal ideal, is that (α) be neither the p_1^{th}, p_2^{th}, ..., nor p_r^{th} power of a principal ideal, where $p_1, p_2, \ldots, p_r$ are the different prime factors of n.*

The problem of determining the lowest power of $\mathfrak{p}$, which is a principal ideal, be reduced, therefore, to that of determining whether a given principal ideal be the p^{th} power of a principal ideal, where p is a prime number.

To determine, therefore, whether (α) be the p^{th} power of a principal ideal, we must multiply α with each one of a system of units,

$$\eta_1, \eta_2, \ldots, \eta_r$$

such that if α be not the p^{th} power of a principal ideal but acquire this property through multiplication by a unit η, then one of the units of this system will be η, and hence one of these products will be the p^{th} power of an integer.

Such a system of units I call a *complete unit system* for the power p. (α), therefore, is or is not the p^{th} power of a principal ideal according as *one* or *none* of the products

$$\alpha, \ \alpha\eta_1, \ \alpha\eta_2, \ \ldots, \ \alpha\eta_r$$

is the p^{th} power of a principal ideal, where $\eta_1, \ldots, \eta_r$ is a complete unit system for the power p. In the construction of such a system of units in the case of a cubic field, we must distinguish two cases, according as Δ is negative or positive, that is, according as among the conjugate fields k, k', k'' there are two or no imaginary fields, and k has respectively one or two fundamental units. Confining ourselves to the case Δ negative and p an odd prime, such a system of units may be constructed for the power p as follows. All units of the field have the form $\eta = \pm\,\varepsilon^m$, where ε is a fundamental unit.

If α be the p^{th} power of an integer or acquire this property through multiplication by a unit, it has the form, when $p \neq 2$,

$$\alpha = \beta^p \varepsilon^l.$$

If we take $\eta = \pm\,\varepsilon^g$, where $g \equiv -l \pmod{p}$, then

$$\alpha\eta = \pm\,\beta^p \varepsilon^{l+g}.$$

The p units

$$\eta = \varepsilon^g \ (\text{or } -\varepsilon^g), \ g \equiv 0, \ 1, \ 2, \ \ldots, \ p-1, \ (\text{mod } p)$$

form a complete unit system for the power p. To obtain such a system we have only to find a unit η, which is not the p^{th} power of a unit, i. e.,

$$\eta = \varepsilon^g \ (\text{or } -\varepsilon^g), \ g \not\equiv 0, \ (\text{mod } p)$$

It is evidently indifferent whether we have $\eta = \varepsilon^g$ or $\eta = -\varepsilon^g$ since p is odd. For the sake of simplicity, we shall take the $+$ sign, though the unit found might have the $-$ sign. $\eta^0 = 1$, η, $\eta^2, \ldots, \eta^{p-1}$ constitute then a complete unit system for the power p, since $0, g, 2g, \ldots, (p-1)\,g$ form a complete remainder system with respect to the modulus p, and hence one of these exponents is $\equiv -l \pmod{p}$. (α), therefore, is or is not the p^{th} power of a principal ideal according as one or none of the products

$$\alpha, \ \alpha\eta, \ \alpha\eta^2, \ \ldots, \ \alpha\eta^{p-1}$$

is the p^{th} power of an integer. When $p=2$, we have merely in addition to the above to take into consideration the multiplication of α by the unit -1, which is unnecessary when p is odd, since then $-\alpha$ is or is not the p^{th} power of an integer according as α is or is not the p^{th} power of an integer. When Δ is positive, the method is similar, account simply being taken of the fact that the field has two fundamental units.

In order to determine whether any one of the above products be the p^{th} power of an integer, we set it equal to $(a\omega_1+b\omega_2+c)^p$, where $\omega_1, \omega_2, 1$ is a basis of k. We then obtain, by equating the coefficients of the corresponding powers of θ on the two sides of the equation, three equations to determine a, b and c. The necessary and sufficient condition for the product under discussion to be the p^{th} power of an integer is that these equations have an integral solution.

By means now of Theorem 4 and the above method for determining whether a given principal ideal is the p^{th} power of a principal ideal, we can determine the lowest power of $\mathfrak{p}$, which is a principal ideal. Let this power be the t^{th}, then

$$\mathfrak{p},\ \mathfrak{p}^2,\ \ldots\ ,\ \mathfrak{p}^{t-1},\ \mathfrak{p}^t \sim (1)$$

are representatives of t different ideal classes, which we denote with

$$A,\ A^2,\ \ldots\ ,\ A^{t-1},\ A^t \sim 1.$$

The class number, h, must now be divisible by t. Let N be the number of ideals whose norms satisfy (1), the unit ideal (1) being included.

If $N < 2t$, we have at once $h=t$.

If, however, $N \nless 2t$, we determine the classes of some of the remaining ideals satisfying 1.

Let $\mathfrak{j}$ be one of them. If we can find a principal ideal (γ) such that $(\gamma)=\mathfrak{p}^r\mathfrak{j}$, then $\mathfrak{j}$ belongs to the class which is reciprocal to that of $\mathfrak{p}^r$, i. e., to $A^{r'}$, where $r' \equiv -r \pmod{t}$. We can then easily determine in which classes lie the different powers of $\mathfrak{j}$, and the products of these powers, with those of $\mathfrak{p}$. If we cannot find such a principal ideal (γ), we must determine the lowest power of $\mathfrak{j}$. which is a principal ideal, exactly as in the case of $\mathfrak{p}$. Let this power be the s^{th}. We must now determine whether $\mathfrak{j}$ lies in any one of the classes $A, A^2, \ldots, A^{t-1}$, It can be shown that it is possible for $\mathfrak{j}$ to lie in one of these classes only when we have $t \equiv 0 \pmod{s}$, in which case, if $\dfrac{t}{s}=t'$, it is possible to have $\mathfrak{j} \sim \mathfrak{p}^{i't'}$, where i' is prime to s. There are, therefore, at most $\phi(s)$ classes, in one of

which $\mathfrak{j}$ may lie. If $s \equiv 0 \pmod{t}$, it would be possible likewise for $\mathfrak{p}$ to lie in one of $\phi(t)$ of the classes $B, B^2, \ldots, B^{s-1}$ represented by the powers of $\mathfrak{j}$. To determine whether $\mathfrak{j}$ lies in the class A^i, we must determine whether the product of $\mathfrak{j}$ and an ideal $\mathfrak{h}$ belonging to the class, which is reciprocal to that of A^i, is a principal ideal. To do this we must find a principal ideal (λ), which is a power of $\mathfrak{j}\mathfrak{h}$, and proceed as already indicated. If $\mathfrak{j}$ belongs to none of the classes $1, A, \ldots, A^{t-1}$, we have st different classes. If $N < 2st$, then $h = st$, and in general, letting, at any point of the reckoning, n be the number of the ideals satisfying 1 whose classes have been determined, k the number of the known classes which have found representatives among these ideals, and K the number of the known classes, from

$$N - n + k < 2K \tag{2}$$

follows

$$h = K.$$

The use of (2) saves much reckoning, as we find from it that we have to determine the classes of only $N' + 1$ of the ideals satisfying (1), where $N = 2N'$, or $2N' + 1$.

The table is arranged as follows: Part I contains all fields defined by the root of an equation of the form

$$x^3 + A_2 x + A_3 = 0,$$

where A_2, A_3 are rational integers less in absolute value than 10. The first column contains a number for purposes of reference; the second, the equation whose root defines the field; the third, the discriminant of the equation; the fourth and fifth, the discriminant of the field; the sixth, the class-number; the seventh, a basis (when no numbers stand here, $\theta^2, \theta, 1$ are to be understood); the eighth, one or more units. When, after a unit η, $\neq k^2$ is placed, this denotes that neither η nor $-\eta$ is the square of a unit. Then follow the factorizations of those rational primes $p < M|\sqrt{\Delta}|$. The factors are arranged according to the magnitude of their norms; i. e., the factor with greatest norm stands second. When $h = 1$, the factors are chosen so that their norms are positive and their product is equal to p without multiplication by a unit, except when p is divisible by the cube of a prime number. In this case, the unit is given with which the cube of the prime must be multiplied to obtain p. These units are designated by * attached to the parentheses in which they stand.

$p = p$ or $(p) = (p)$ means that p is unfactorable. When $\equiv$ and a number stand in the seventh column, it signifies that this field is identical with the field designated by the number.

Part II contains all fields defined by a root of a cubic equation of the form

$$A_0 x^3 + A_1 x^2 + A_2 x + A_3 = 0,$$

where A_0, A_1, A_2, A_3 are rational integers less in absolute value than 3, with the exception of those equations of this form which are found in Part I or are transformable into one of those of Part I by the substitution $x : x - \frac{A_1}{3A_0}$. In the cases also where one of these equations is transformable into another of the same form by a linear substitution, one only of the two is given.

Part II is arranged like Part I, with the exception that in **12**, **16**, **17**, **18**, **19**, since the roots of these equations are not integers, the equations written immediately under have been used to define these fields. In each case, this equation is obtained from the original one by the substitution $x : \frac{x}{2}$.

A fuller discussion of the methods employed in the calculation of this table, with numerical examples, will be found in "Tafel der Klassenanzahlen für kubische Zahlenkörper," published by the author of this article as dissertation.

See Hilbert, "Bericht über die Theorie der algebraischen Zahlkörper," §§ 11, 24. Jahresbericht der deutschen Mathematiker-Vereinigung, Vierter Band, 1894-95.

Minkowski, "Geometrie der Zahlen."

Woronoj, "The algebraic integers, which are functions of a root of an equation of the 3d degree." (Translation of the Russian title.)

PRINCETON, N. J.

PART I.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
1	$x^3+1=0$	red.						1 :
2	$x^3+x+1=0$	-31	-31	-31	1	......	$\theta,\ \theta+1$	$2:2\ =2\ \ 3=(-\theta+1)(\theta^2+\theta+2)\ \ 5=5$
3	$x^3-x+1=0$	-23	-23	-23	1	......	$\theta,\ \theta-1,\ \theta+1$	$3:2\ =2\ \ 3=3$
4	$x^3+2=0$	$-2^2.3^3$	$-2^2.3^3$	-108	1	. ...	$\theta+1,\ \theta^2-\theta+1$	$4:2\ =(-\theta)^3\ 3=(-\theta+1)^3(-\theta-1)^*\ \ 5=(\theta^2+1)(-\theta^2-2\theta+1)\ \ 7=7$
5	$x^3+x+2=0$	red.						5 :
6	$x^3-x+2=0$	$-2^3.13$	$-2^3.13$	-104	1	......	$\theta^2+\theta-1$	$6:2\ =(-\theta+1)^2(\theta^2-2)\ \ 3=3\ \ 5=5\ \ 7=7$
7	$x^3+2x+2=0$	$-2^2.5.7$	$-2^2.5.7$	-140	1	......	$\theta+1$	$7\begin{cases}2\ =(-\theta)^3(\theta^2-\theta+3)^*\ \ 3=3\ \ 5=(-\theta+1)^2(\theta^2+1)\\ 7\ =(2\theta^2-1)^2(52\theta^2-40\theta+135)\end{cases}$
8	$x^3-2x+2=0$	$-2^2.19$	$-2^2.19$	-76	1	......	$\theta-1,\ \theta^2+\theta-1$	$8:2\ =(-\theta)^3(\theta^2+\theta-1)^*\ \ 3=(-\theta-1)(\theta^2-\theta-1)\ \ 5=5\ \ 7=7$
9	$x^3+2x+1=0$	-59	-59	-59	1	......	θ	$9:2\ =(\theta+1)(\theta^2-\theta+3)\ \ 3=3\ \ 5=5\ \ 7=7$
10	$x^3-2x+1=0$	red.						10 :
11	$x^3+3=0$	-3^5	-3^5	-243	1	......	θ^2-2	$11\begin{cases}2\ =(-\theta-1)(\theta^2-\theta+1)\ \ 3=(-\theta)^3\ \ 5=(\theta+2)(\theta^2-2\theta+4)\ \ 7=7\\ 11=(-\theta+2)(\theta^2+2\theta+4)\ \ 13=13\end{cases}$
12	$x^3+x+3=0$	-13.19	$-13.\ 19$	-247	1	......	$\theta+1$	$12:2\ =2\ \ 3=(-\theta)(\theta^2+1)\ \ 5=(-\theta+1)(\theta^2+\theta+2)\ \ 7=(\theta+2)(\theta^2-2\theta+5)$
13	$x^3-x+3=0$	-239	-239	-239	1	.....	$\theta^2+\theta-1$	$13:2\ =2\ \ 3=(-\theta)(-\theta+1)(-\theta-1)\ \ 5=5$
14	$x^3+2x+3=0$	red.						14 :
15	$x^3-2x+3=0$	-211	-211	-211	1	......	$\theta^2-2\theta+2$	$15:2\ =(-\theta+1)(\theta^2+\theta-1)\ \ 3=(-\theta)(\theta^2-2)$
16	$x^3+3x+3=0$	$-3^3.13$	$-3^3.13$	-351	1	......	$\theta+1$	$16\begin{cases}2\ =2\ \ 3=(-\theta)^3(\theta^2-\theta+4)^*\ \ 5=5\ \ 7=(-\theta+1)(\theta^2+\theta+4)\\ 11=(\theta+2)(2\theta+1)(-3\theta^2+2\theta-11)\ \ 13=(2\theta^2-\theta+7)^2(2\theta^2-\theta-2)\\ 17=(-\theta+2)(\theta^2+2\theta+7)\end{cases}$
17	$x^3-3x+3=0$	$-3^3.5$	$-3^3.5$	-135	1	......	$\theta-1,\ \theta+2$	$17:2\ =2\ \ 3=(-\theta)^3(\theta^2+\theta-2)^*\ \ 5=(-\theta-1)^2(-\theta+2)$
18	$x^3+3x+1=0$	$-3^3.5$	$-3^3.5$	-135	1	......	$\theta,\ 3\theta+1,\ \theta^2+3$	$18:2\ =2\ \ 3=(-\theta-1)^3(3\theta^2-\theta+9)^*\ \ 5=(2\theta+1)^2(12\theta^2-4\theta+37)$
19	$x^3-3x+1=0$	3^4	3^4	81	1	......	$\theta,\ \theta-1,\ \theta+2$	$19:2\ =2\ \ 3=(-\theta-1)^3(\theta^2-\theta-1)^*\ \ 5=5\ \ 7=7$
20	$x^3+3x+2=0$	$-2^3.3^3$	$-2^3.3^3$	-216	1	......	$-\theta^2+\theta+1$	$20\begin{cases}2\ =(\theta+1)^2(3\theta^2-2\theta+10)\ \ 3=(-2\theta-1)^3(104\theta^2-62\theta+349)^*\ \ 5=5\\ 7\ =7\ \ 11=11\end{cases}$
21	$x^3-3x+2=0$	red.						21 :
22	$x^3+4=0$	$-2^4.3^3$	$-2^2.3^3$	-108	1	$\equiv 4,\ \frac{\theta^2}{2},\ \theta,\ 1$	$\frac{\theta^2-2}{2}$	$22:2\ =\left(\frac{\theta^2}{2}\right)^3\ 3=(\theta+1)^3(-2\theta^2+3\theta-5)^*$
23	$x^3+x+4=0$	$-2^2.109$	$-2^2.109$	-436	1	......	$21\theta^2-29\theta+61$	$23\begin{cases}2\ =(-\theta-1)^2(2\theta^2-3\theta+6)\ \ 3=(\theta^2-\theta+3)(-\theta^2-\theta+1)\ \ 5=5\\ 7\ =(2\theta+3)(4\theta^2-6\theta+13)\ \ 11=11\ \ 13=(\theta^2-\theta+1)(\theta^2-3\theta-3)\\ 17=(2\theta^2-2\theta+5)(-2\theta^2-6\theta-3)\end{cases}$
24	$x^3-x+4=0$	$-2^2.107$	-107	-107	1	$\frac{\theta^2+\theta}{2},\ \theta,\ 1$	$5\theta^2-9\theta+11\pm k^2$ $\theta^2-\theta-5,\ \frac{\theta^2-\theta+2}{2}$	$24\begin{cases}2\ =(\theta+2)(\theta^2-2\theta+3)\ \ 3=3\ \ 5=(2\theta^2-4\theta+5)(2\theta^2+4\theta+1)\\ 7\ =(2\theta^2+2\theta-3)(-6\theta^2+10\theta-13)\ \ 11=11\ \ 13=13\end{cases}$
25	$x^3+2x+4=0$	$-2^4.29$	$-2^2.29$	-116	1	$\frac{\theta^2}{2},\ \theta,\ 1$	$\theta+1$	$25\begin{cases}2\ =\left(\frac{\theta^2+2}{2}\right)^2\left(\frac{\theta^2}{2}\right)\ \ 3=3\ \ 5=5\ \ 7=(-\theta+1)(\theta^2+\theta+3)\ \ 11=11\ \ 13=13\\ 17=(\theta^2+1)(\theta^2-4\theta+1)\ \ 19=(-2\theta-3)(\theta^2+3)(-\theta^2+2\theta-3)\end{cases}$

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
26	$x^3-2x+4=0$	red.						26 :
27	$x^3+3x+4=0$	red.						27 :
28	$x^3-3x+4=0$	$-2^2 . 3^4$	$-2^2 . 3^4$	-324	1	......	$\theta^2-\theta-7$	28 $\begin{cases} 2=(-\theta+1)^2(-\theta-2)\ \ 3=(-\theta^2-\theta+3)^3(-5\theta^2+11\theta-9)^*\ \ 5=5 \\ 7=(2\theta^2+2\theta-5)(2\theta^2-6\theta+5)\ \ 11=(\theta^2+\theta-1)(-\theta^2-3\theta+5)\ \ 13=13 \\ 17=17 \end{cases}$
29	$x^3+4x+4=0$	$-2^4 . 43$	$-2^2 . 43$	-172	1	$\frac{\theta^2}{2}, \theta, 1$	$\theta+1,\ \theta^2-\theta+5$	29 $\begin{cases} 2=\left(\frac{\theta^2+4}{2}\right)^3(\theta+1)^*\ \ 3=\left(\frac{-\theta^2-2}{2}\right)\left(\frac{-\theta^2+2\theta-2}{2}\right) \\ 5=\left(\frac{\theta^2+6}{2}\right)\left(\frac{-\theta^2-2\theta+2}{2}\right)\ \ 7=\left(\frac{-\theta^2+2}{2}\right)\left(\frac{3\theta^2-2\theta+18}{2}\right)\ \ 11=11 \\ 13=13\ \ 17=17 \end{cases}$
30	$x^3-4x+4=0$	$-2^4 . 11$	$-2^2 . 11$	-44	1	$\frac{\theta^2}{2}, \theta, 1$	$\theta-1$	30 $\begin{cases} 2=\left(\frac{\theta^2}{2}\right)^3(-\theta^2-2\theta+1)^*\ \ 3=3\ \ 5=5\ \ 7=(-\theta-1)(\theta^2-\theta-3) \\ 11=\theta^2-5)^2(2\theta^2-4\theta+3)\ \ 13=(\theta^2-3)(-\theta^2-4\theta+1) \end{cases}$
31	$x^3+4x+1=0$	-283	-283	-283	2	......	$\theta\pm k^2,\ 4\theta+1$	31 $\begin{cases} (2)=(2,\theta+1)(2,\theta^2+\theta+1)\ \ (3)=(3,\theta-1)(3,\theta^2+\theta-1) \\ (5)=(5,\theta+2)(5,\theta^2-2\theta+3)\ \ (7)=(7)\ \ (11)=(11)\ \ (13)=(13) \end{cases}$
32	$x^3-4x+1=0$	229	229	229	1	......	$\theta,\ \theta+2,\ \theta-2$	32 $\begin{cases} 2=(\theta-1)(\theta^2+\theta-3)\ \ 3=3\ \ 5=5\ \ 7=(2\theta-1)(4\theta^2+2\theta-15) \\ 11=11\ \ 13=(2\theta-3)(4\theta^2+6\theta-7) \end{cases}$
33	$x^3+4x+2=0$	$-2^2.7.13$	$-2^2.7.13$	-364	1	......	$2\theta+1$	33 $\begin{cases} 2=(-\theta)^3(4\theta^2-2\theta+17)^*\ \ 3=(\theta+1)(\theta^2-\theta+5)\ \ 5=5 \\ 7=(-\theta+1)^2(\theta^2+3)\ \ 11=(15\theta^2-7\theta+63)(4\theta^2-9\theta-5) \\ 13=(3\theta^2-\theta+13)^2(\theta^2-6\theta-3)\ \ 17=(-2\theta^2+\theta-7)(-3\theta^2+\theta-1) \end{cases}$
34	$x^3-4x+2=0$	$2^2 . 37$	$2^2 . 37$	148	1	......	$\theta-1,\ -2\theta+1$	34 : $2=(-\theta)^3(4\theta^2+2\theta-15)^*\ \ 3=3\ \ 5=(-\theta-1)(\theta^2-\theta-3)\ \ 7=7\ \ 11=11$
35	$x^3+4x+3=0$	-499	-499	-499	1	......	$3\theta+2$	35 : $2=(\theta+1)(\theta^2-\theta+5)\ \ 3=(-\theta)(\theta^2+4)\ \ 5=5$
36	$x^3-4x+3=0$	red.						36 :
37	$x^3+5=0$	$-3^3 . 5^2$	$-3^3 . 5^2$	-675	1	......	$2\theta^2+4\theta+1$	37 $\begin{cases} 2=(\theta^2-2\theta+3)(\theta^2+\theta-1)\ \ 3=(\theta+2)^3(14\theta^2-24\theta+41)^*\ \ 5=(-\theta)^3 \\ 7=7\ \ 11=(2\theta^2-3\theta+6)(-3\theta^2-2\theta+6)\ \ 13=(-\theta+2)(\theta^2+2\theta+4) \\ 17=(\theta^2-2)(2\theta^2-5\theta+4)\ \ 19=19\ \ 23=(2\theta^2-4\theta+7)(2\theta^2+8\theta+9) \end{cases}$
38	$x^3+x+5=0$	$-7 . 97$	$-7 . 97$	-679	1	......	$-4\theta^2+6\theta-13$	38 $\begin{cases} 2=2\ \ 3=(-\theta-1)(\theta^2-\theta+2)\ \ 5=(-\theta)(\theta+2)(\theta^2-\theta+3) \\ 7=(-\theta^2+2\theta-4)^2(2\theta^2+\theta-3)\ \ 11=11\ \ 13=13\ \ 17=17\ \ 19=19 \\ 23=(\theta^2-3)(-4\theta^2+5\theta-16) \end{cases}$
39	$x^3-x+5=0$	$-11 . 61$	$-11 . 61$	-671	1	......	$\theta+2$	39 $\begin{cases} 2=2\ \ 3=3\ \ 5=(-\theta)(\theta+1)(\theta-1)\ \ 7=7\ \ 11=(-\theta+2)^2(\theta^2+\theta-1) \\ 13=(\theta^2-3)(2\theta^2-5\theta+4)\ \ 17=17\ \ 19=(\theta+3)(\theta^2-3\theta+8) \\ 23=(\theta^2-2)(\theta^2-5\theta+1) \end{cases}$
40	$x^3+2x+5=0$	$-7 . 101$	$-7 . 101$	-707	1	......	$3\theta+4$	40 $\begin{cases} 2=(-\theta-1)(\theta^2-\theta+3)\ \ 3=3\ \ 5=(-\theta)(\theta^2+2) \\ 7=(\theta+2)^2(2\theta^2-3\theta+8)\ \ 11=(2\theta+3)(4\theta^2-6\theta+17) \\ 13=(\theta^2-2\theta+4)(2\theta+3\theta+2)\ \ 17=(-\theta+2)(\theta^2+2\theta+6) \\ 19=(2\theta^2-2\theta+7)(-2\theta^2-6\theta-3)\ \ 23=23 \end{cases}$

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
41	$x^3-2x+5=0$	-643	-643	-643	2	......	$\theta+2\pm k^2$	41 : $(2)=(2,\theta+1)(2,\theta^2+\theta+1)$ $(3)=(3,\theta+1)(3,\theta^2-\theta-1)$ $(5)=(-\theta)(\theta^2-2)$ $(7)=(7)$
42	$x^3+3x+5=0$	$-3^3\,.\,29$	$-3\,.\,29$	-87	1	$\frac{\theta^2+\theta+1}{3},\theta,1$	$\theta+1,\frac{\theta^2+\theta+1}{3}$	42 : $2=2$ $3=\left(\frac{\theta^2-2\theta-2}{3}\right)^2\left(\frac{4\theta^2-5\theta+19}{3}\right)$ $5=(-\theta)(\theta^2+3)$ $7=7$
43	$x^3-3x+5=0$	$-3^4\,.\,7$	$-3^4\,.\,7$	-567	1	......	$\theta^2-2\theta+2$	43 $\begin{cases}2=2\ \ 3=(-\theta+1)^3(\theta^2+\theta-3)^*\ \ 5=(-\theta)(\theta^2-3)\\ 7=(-\theta-1)^2(-\theta+2)\ \ 11=11\ \ 13=(-\theta-3)(-\theta^2+3\theta-6)\\ 17=(-\theta^2-\theta+4)(2\theta^2-\theta+3)\ \ 19=(\theta^2+\theta-1)(-\theta^2-4\theta+6)\\ 23=(-\theta+3)(\theta^2-2)(-\theta^2-2\theta+2)\end{cases}$
44	$x^3+4x+5=0$	red.						44 :
45	$x^3-4x+5=0$	-419	-419	-419	1	......	$2\theta^2-5\theta+4$	45 $\begin{cases}2=(-\theta+1)(\theta^2+\theta-3)\ \ 3=3\ \ 5=(-\theta)(\theta+2)(\theta-2)\ \ 7=7\\ 11=(-\theta^2-2\theta+2)(2\theta^2-\theta-2)\ \ 13=13\ \ 17=(\theta^2-2)(-2\theta^2-5\theta+4)\end{cases}$
46	$x^3+5x+5=0$	$-5^2\,.\,47$	$-5^2\,.\,47$	-1175	1	......	$\theta+1$	46 $\begin{cases}2=2\ \ 3=3\ \ 5=(-\theta)^3(\theta^2-\theta+6)^*\ \ 7=7\ \ 11=(-\theta+1)(\theta^2+\theta+6)\\ 13=(\theta+2)(\theta^2-2\theta+9)\ \ 17=17\ \ 19=(-2\theta-1)(4\theta^2-2\theta+21)\\ 23=(-\theta+2)(\theta^2+2\theta+9)\ \ 29=(\theta^2+4)(\theta^2-5\theta+1)\\ 31=(\theta^2+\theta+1)(5\theta^2-6\theta+26)\end{cases}$
47	$x^3-5x+5=0$	$-5^2\,.\,7$	$-5^2\,.\,7$	-175	1	......	$\theta-1$	47 $\begin{cases}2=2\ \ 3=(-\theta+2)(\theta^2+2\theta-1)\ \ 5=(-\theta)^3(\theta^2+\theta-4)^*\\ 7=(-\theta-2)^2(\theta^2-3\theta+3)\ \ 11=11\end{cases}$
48	$x^3+5x+1=0$	$-17\,.\,31$	$-17\,.\,31$	-527	1	......	θ	48 : $2=2$ $3=3$ $5=(\theta+1)(\theta^2-\theta+6)$
49	$x^3-5x+1=0$	$11\,.\,43$	$11\,.\,43$	473	1	......	θ	49 : $2=2$ $3=(\theta-1)(\theta^2+\theta-4)$
50	$x^3+5x+2=0$	$-2^5\,.\,19$	$-2^3\,.\,19$	-152	1	$\frac{\theta^2+\theta}{2},\theta,1$	......	50 : $2=\left(\frac{\theta^2-\theta+4}{2}\right)^2(-\theta)$ $3=3$
51	$x^3-5x+2=0$	red.						51 :
52	$x^3+5x+3=0$	-743	-743	-743	1	......	......	52 $\begin{cases}2=2\ \ 3=(-\theta)(\theta+1)(2\theta^2-\theta+11)\ \ 5=(2\theta^2-\theta+10)(\theta^2-2\theta-1)\\ 7=(\theta^2-\theta+5)(\theta^2+2\theta+2)\end{cases}$
53	$x^3-5x+3=0$	257	257	257	1	......	$\theta-1$	53 : $2=2$ $3=(-\theta)(\theta^2-5)$
54	$x^3+5x+4=0$	$-2^2\,.\,233$	$-2^2\,.\,233$	-932	1	......	......	54 $\begin{cases}2=(\theta+1)^2(4\theta^2-3\theta+22)\ \ 3=3\ \ 5=(\theta^2-\theta+5)(\theta^2+\theta+1)\\ 7=(2\theta^2-1)(22\theta^2-16\theta+121)\end{cases}$
55	$x^3-5x+4=0$	red.						55 :
56	$x^3+6=0$	$-2^2\,.\,3^5$	$-2^2\,.\,3^5$	-972	1	......	$3\theta^2+6\theta+1$	56 $\begin{cases}2=(\theta+2)^3(33\theta^2-60\theta+109)^*\ \ 3=(\theta^2-2\theta+3)^3(3\theta^2+6\theta+1)^*\\ 5=(-\theta-1)(\theta^2-\theta+1)\ \ 7=(-\theta+1)(2\theta^2-4\theta+7)(\theta^2-\theta-5)\end{cases}$
57	$x^3+x+6=0$	$-2^4\,.\,61$	$-2^2\,.\,61$	-244	1	$\frac{\theta^2+\theta}{2},\theta,1$	......	57 : $2=\left(\frac{\theta^2-\theta+4}{2}\right)^2\left(\frac{-\theta^2-3\theta-2}{2}\right)$ $3=\left(\frac{3\theta^2-5\theta+12}{2}\right)\left(\frac{-\theta^2+3\theta+8}{2}\right)$
58	$x^3-x+6=0$	red.						58 :
59	$x^3+2x+6=0$	$-2^2\,.\,251$	$-2^2\,.\,251$	-1004	1	......	$7\theta^2+15\theta+7$	59 : $2=(2\theta^2-3\theta+8)^3(7\theta^2+15\theta+7)^*$ $3=(-\theta-1)(2\theta+3)(9\theta^2-13\theta+37)$
60	$x^3-2x+6=0$	$-2^2.5.47$	$-2^2.5.47$	-940	1	......	$3\theta^2+7\theta+1$	60 $\begin{cases}2=(-\theta-2)^3(28\theta^2-61\theta+77)^*\ \ 3=(\theta^2-2\theta+3)(-\theta^2+5)\\ 5=(2\theta^2-4\theta+5)^2(4\theta^2-19)\ \ 7=(-\theta-1)(\theta^2-\theta-1)\end{cases}$

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
61	$x^3+3x+6=0$	$-2^3.3^3.5$	$-2^3.3^3.5$	-1080	1	......	......	61 $\begin{cases} 2 = (-\theta-1)^2(3\theta^2-4\theta+14) \;\; 3=(-\theta^2+\theta+3)^3(2473\theta^2-3185\theta+11521)^* \\ 5 = (\theta^2-\theta+5)(-\theta^2-\theta+1) \;\; 7=(2\theta^2+8\theta+7)(62\theta^2-80\theta+289) \end{cases}$
62	$x^3-3x+6=0$	$-2^5.3^3$	$-2^3.3^3$	-216	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	$\frac{-\theta^2+\theta+8}{2}$	62 : $2 = \left(\frac{-\theta^2-\theta+4}{2}\right)^2(\theta^2-2\theta+2) \;\; 3=\left(\frac{\theta^2-\theta}{2}\right)^3\left(\frac{-\theta^2+\theta+8}{2}\right)^*$
63	$x^3+4x+6=0$	$-2^2.307$	$-2^2.307$	-1228	3	......	$\theta+1\mp k^3$	63 : $(2)=(2,\theta)^3 \;\; (3)=(3,\theta)(3,\theta^2+4) \;\; (5)=(5,\theta+2)(5,\theta^2-2\theta+8) \;\; (7)=(7)$
64	$x^3-4x+6=0$	$-2^2.179$	$-2^2.179$	-716	1	......	$40\theta^2-101\theta+95$	64 $\begin{cases} 2 = (-\theta^2-\theta+4)^3(40\theta^2-101\theta+95)^* \;\; 3=(-\theta+1)(\theta^2-3\theta+3)(-2\theta-5) \\ 5 = 5 \;\; 7=(\theta^2+2\theta-1)(\theta^2-4\theta+5) \end{cases}$
65	$x^3+5x+6=0$	red.						65 :
66	$x^3-5x+6=0$	$-2^3.59$	$-2^3.59$	-472	1	......	......	66 $\begin{cases} 2 = (-\theta+1)^2(-\theta^2-2\theta+2) \;\; 3=(\theta^2-3\theta+3)(\theta^2+3\theta+1) \\ 5 = (-\theta^2-\theta+5)(\theta^2-\theta+1) \end{cases}$
67	$x^3+6x+6=0$	$-2^2.3^3.17$	$-2^2.3^3.17$	-1836	3	......	$\theta+1\mp k^3$	67 $\begin{cases} (2) = (2,\theta)^3 \;\; (3)=(3,\theta)^3 \;\; (5)=(5) \;\; (7)=(7,\theta+2)(7,\theta^2-2\theta+3) \\ (11)=(11) \end{cases}$
68	$x^3-6x+6=0$	$-2^2.3^3$	$-2^2.3^3$	-108	1	$\equiv 4$	......	68 :
69	$x^3+6x+1=0$	$-3^4.11$	$-3^4.11$	-891	3	......	$\theta\mp k^3 \quad 6\theta+1$	69 $\begin{cases} (2)=(2,\theta+1)(2,\theta^2+\theta+1) \;\; (3)=(3,\theta+1)^3 \;\; (5)=(5) \\ (7)=(7,\theta-2)(7,\theta^2+2\theta+3) \end{cases}$
70	$x^3-6x+1=0$	$3^3.31$	$3^3.31$	837	1	......	θ	70 $\begin{cases} 2 = (-\theta^2-2\theta+1)(-\theta^2+\theta+3) \;\; 3=(\theta-2)^3(6\theta^2+14\theta-3)^* \\ 5 = (-\theta-2)(\theta^2-2\theta-2) \end{cases}$
71	$x^3+6x+2=0$	$-2^2.3^5$	$-2^2.3^3$	-108	1	$\equiv 4, \frac{\theta^2+\theta+1}{3},$	......	71 :
72	$x^3-6x+2=0$	$2^2.3^3.7$	$2^2.3^3.7$	756	1	$\theta-1, 1$	$3\theta-1$	72 : $2 = (-\theta)^3(9\theta^2+3\theta-53)^* \;\; 3=(\theta-1)^3(2\theta^2+\theta-11)^* \;\; 5=5$
73	$x^3+6x+3=0$	$-3^3.41$	$-3^3.41$	-1107	2	......	$2\theta+1\mp k^2$	73 $\begin{cases} (2)=(2,\theta+1)(2,\theta^2+\theta+1) \;\; (3)=(-\theta)^3 \;\; (5)=(5,\theta-1)(5,\theta^2+\theta+2) \\ (7)=(7,\theta+3)(7,\theta^2-3\theta+1) \end{cases}$
74	$x^3-6x+3=0$	$3^3.23$	$3^3.23$	621	1	......	$\theta-2$	74 : $2 = (\theta-1)(\theta^2+\theta-5) \;\; 3=(-\theta)^3(4\theta^2+2\theta-23)^* \;\; 5=5$
75	$x^3+6x+4=0$	$-2^4.3^4$	$-2^2.3^4$	-324	1	$\frac{\theta^2}{2}, \theta, 1$	$\theta^2-\theta-1$	75 : $2 = \left(\frac{\theta^2+6}{2}\right)^2\left(\frac{\theta^2}{2}\right) \;\; 3=(\theta+1)^3(8\theta^2-5\theta+51)^* \;\; 5=5$
76	$x^3-6x+4=0$	red.						76 :
77	$x^3+6x+5=0$	$-3^4.19$	$-3^4.19$	-1539	1	......	$2\theta^2-34\theta-27$	77 $\begin{cases} 2 = (\theta+1)(\theta^2-\theta+7) \;\; 3=(4\theta^2-3\theta+26)^3(2\theta^2-34\theta-27)^* \\ 5 = (-\theta)(21\theta^2-16\theta+138)(-4\theta-3) \;\; 7=7 \;\; 11=11 \end{cases}$
78	$x^3-6x+5=0$	red.						78 :
79	$x^3+7=0$	$-3^3.7^2$	$-3^3.7^2$	-1323	3	......	$\theta+2\mp k^3$	79 $\begin{cases} (2)=(2,\theta+1)(2,\theta^2+\theta+1) \;\; (3)=(3,\theta+1)^3 \\ (5)=(5,\theta-2)(5,\theta^2+2\theta+4) \;\; (7)=(-\theta)^3 \end{cases}$
80	$x^3+x+7=0$	-1327	-1327	-1327	1	......	......	80 : $2 = 2 \;\; 3=(\theta+2)(\theta^2-2\theta+5) \;\; 5=(-\theta-1)(\theta^2-\theta+2) \;\; 7=(-\theta)(\theta^2+1$
81	$x^3-x+7=0$	-1319	-1319	-1319	1	......	$\theta+2$	81 : $2 = 2 \;\; 3=3 \;\; 5=5 \;\; 7=(-\theta)(-\theta+1)(-\theta-1)$
82	$x^3+2x+7=0$	-5.271	-5.271	-1355	1	......	......	82 $\begin{cases} 2 = (-2\theta^2+3\theta-9)(\theta^2+\theta-1) \;\; 3=3 \;\; 5=(2\theta+3)^2(28\theta^2-44\theta+125) \\ 7 = (-\theta)(\theta^2+2) \end{cases}$
83	$x^3-2x+7=0$	-1291	-1291	-1291	1	......	......	83 $\begin{cases} 2 = (\theta^2-2\theta+3)(-\theta^2-\theta+3) \;\; 3=(-\theta-2)(\theta^2-2\theta+2) \;\; 5=5 \\ 7 = (-\theta)(-2\theta^2-\theta+8)(9\theta^2-20\theta+28) \end{cases}$

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
84	$x^3+3x+7=0$	$-3^3.53$	$-3^3.53$	-1431	1	.....	$-\theta^2+2$	$84:2=2\ \ 3=(-\theta-1)^3(5\theta^2-7\theta+25)^*\ \ 5=5\ \ 7=(-\theta)(\theta+2)(\theta^2-\theta+5)$
85	$x^3-3x+7=0$	$-3^5.5$	$-3^3.5$	-135	1	$\frac{\theta^2-\theta+1}{3}, \theta, 1$	$\frac{\theta^2-\theta+1}{3}$	$85:2=2\ \ 3=\left(\frac{\theta^2+2\theta+1}{3}\right)^3\left(\frac{\theta^2-\theta-8}{3}\right)^*$
86	$x^3+4x+7=0$	-1579	-1579	-1579	1	......	......	$86\begin{cases}2=(-\theta-1)(\theta^2-\theta+5)\ \ 3=(4\theta^2-5\theta+22)(\theta^2-2\theta-4)\ \ 5=5\\ 7=(-\theta)(\theta^2+4)\ \ 11=11\end{cases}$
87	$x^3-4x+7=0$	-11.97	-11.97	-1067	1	......	......	$87\begin{cases}2=(-\theta^2-\theta+4)(-\theta^2+3\theta-3)\ \ 3=3\ \ 5=(-\theta^2+2\theta-2)(2\theta^2+3\theta-6)\\ 7=(-\theta)(-\theta+2)(-\theta-2)\end{cases}$
88	$x^3+5x+7=0$	-1823	-1823	-1823	2	......	$\theta+1\pm k^2$	$88\begin{cases}(2)=(2)\ \ (3)=(3)\ \ (5)=(5,\theta-2)(5,\theta^2+2\theta-1)\\ (7)=(-\theta)(7,\theta-3)(7,\theta+3)\ \ (11)=(\theta+2)(11,\theta+4)(11,\theta+5)\end{cases}$
89	$x^3-5x+7=0$	-823	-823	-823	1	......	......	$89\begin{cases}2=2\ \ 3=(-\theta+1)(\theta^2+\theta-4)\ \ 5=(-\theta+2)(\theta^2+2\theta-1)\\ 7=(-\theta)(\theta^2-5)\end{cases}$
90	$x^3+6x+7=0$	red.						90 :
91	$x^3-6x+7=0$	$-3^3.17$	$-3^3.17$	-459	1	......	$2\theta^2+2\theta-11$	$91\begin{cases}2=(-\theta+1)(\theta^2+\theta-5)\ \ 3=(-\theta+2)^3(2\theta^2+2\theta-11)^*\ \ 5=5\\ 7=(-\theta)(\theta^2-6)\ \ 11=(-\theta-2)(2\theta^2+4\theta-5)(3\theta^2-8\theta+6)\ \ 13=13\end{cases}$
92	$x^3+7x+7=0$	$-5.7^2.11$	$-5.7^2.11$	-2695	3	......	$\theta+1\pm k^3$	$92\begin{cases}(2)=(2)\ \ (3)=(3,\theta-1)(3,\theta^2+\theta-1)\ \ (5)=(5,\theta-1)^2(5,\theta+2)\\ (7)=(-\theta)^3\ \ (11)=(11,\theta-4)^2(11,\theta-3)\ \ (13)=(13)\end{cases}$
93	$x^3-7x+7=0$	7^2	7^2	49	1	......	......	93 :
94	$x^3+7x+1=0$	-1399	-1399	-1399	2	......	$\theta\pm k^2$	$94:(2)=(2)\ \ (3)=(3,\theta-1)(3,\theta^2+\theta+2)\ \ (5)=(5)\ \ (7)=(\theta+1)(7,\theta+2)(7,\theta-3)$
95	$x^3-7x+1=0$	5.269	5.269	1345	1	..... .	θ	$95:2=2\ \ 3=3\ \ 5=(\theta-2)^2(\theta^2+3\theta+1)\ \ 7=(-\theta-1)(-\theta-2)(-\theta+3)$
96	$x^3+7x+2=0$	$-2^3.5.37$	$-2^3.5.37$	-1480	1	......	......	$96\begin{cases}2=(2\theta^2-3\theta-1)^2(5501\theta^2-1554\theta+38946)\ \ 3=(14\theta^2-4\theta+99)(2\theta^2+4\theta+1)\\ 5=(25\theta^2-7\theta+177)^2(24\theta^2-18\theta-7)\ \ 7=7\end{cases}$
97	$x^3-7x+2=0$	$2^4.79$	$2^2.79$	316	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	$\frac{\theta^2-\theta}{2}, 3\theta^2+\theta-2$	$97:2=\left(\frac{\theta^2+\theta}{2}\right)^2\left(\frac{-17\theta^2-5\theta+118}{2}\right)\ \ 3=3$
98	$x^3+7x+3=0$	$-5.17.19$	$-5.17.19$	-1615	1	......	......	$98\begin{cases}2=2\ \ 3=(-\theta)(\theta^2+7)\ \ 5=(\theta+1)^2(2\theta^2-\theta+14)\\ 7=7\ \ 11=(-\theta+1)(\theta^2+\theta+8)\end{cases}$
99	$x^3-7x+3=0$	1129	1129	1129	1	......	θ^2-8	$99:2=2\ \ 3=(-\theta)(\theta-1)(-2\theta^2-\theta+13)\ \ 5=5\ \ 7=7$
100	$x^3+7x+4=0$	$-2^2.11.41$	-11.41	-451	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	......	$100:2=\left(\frac{\theta^2-\theta+8}{2}\right)(\theta+1)\ \ 3=\left(\frac{\theta^2-\theta+6}{2}\right)\left(\frac{\theta^2+\theta+2}{2}\right)\ \ 5=5$
101	$x^3-7x+4=0$	$2^2.5.47$	$2^2.5.47$	940	1	......	$-2\theta^2+6\theta-3$	$101:2=(\theta-1)^2(-2\theta^2-\theta+14)\ \ 3=3\ \ 5=(2\theta-1)^2(-20\theta^2-12\theta+133)$
102	$x^3+7x+5=0$	-23.89	-23.89	-2047	1	......	......	$102\begin{cases}2=2\ \ 3=(\theta+1)(\theta^2-\theta+8)\ \ 5=(-\theta)(\theta^2+7)\ \ 7=7\\ 11=(3\theta^2-2\theta+23)(-2\theta^2+\theta+2)\end{cases}$
103	$x^3-7x+5=0$	17.41	17.41	697	1	......	$\theta-2$	$103:2=2\ \ 3=3\ \ 5=(-\theta)(\theta^2-7)$
104	$x^3+7x+6=0$	$-2^3.293$	$-2^3.293$	-2344	1	......	......	$104\begin{cases}2=(\theta+1)^2(5\theta^2-4\theta+38)\ \ 3=(-\theta^2+3\theta+3)(19\theta^2-15\theta+145)\ \ 5=5\\ 7=(\theta^2-\theta+7)(7\theta^2-11\theta-13)\ \ 11=11\ \ 13=13\end{cases}$
105	$x^3-7x+6=0$	red.						105 :
106	$x^3+8=0$	red.						106 :

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
107	$x^3+x+8=0$	$-2^2.433$	$-2^2.433$	-1732	2	......	$6\theta+11 \mp k^2$	107 $\begin{cases} (2)=(\theta+2)(2,\theta-1)^2 \ (3)=(3,\theta+1)(3,\theta^2-\theta+2) \\ (5)=(5,\theta-1)(5,\theta^2+\theta+2) \ (7)=(7) \end{cases}$
108	$x^3-x+8=0$	$-2^2.431$	-431	-431	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	$6\theta-11$	108 : $2=(-\theta-2)(\theta^2-2\theta+3)$ $3=3$ $5=5$
109	$x^3+2x+8=0$	$-2^5.5.11$	$-2^3.5.11$	-440	1	$\frac{\theta^2}{2}, \theta, 1$	......	109 : $2=\left(\frac{\theta^2-2\theta-6}{2}\right)^2\left(\frac{\theta^2+4\theta+4}{2}\right)$ $3=3$ $5=(-\theta-1)^2(\theta^2-2\theta+5)$
110	$x^3-2x+8=0$	$-2^5.53$	$-2^3.53$	-424	1	$\frac{\theta^2}{2}, \theta, 1$	......	110 : $2=\left(\frac{\theta^2-2\theta+2}{2}\right)^2\left(\frac{\theta^2+4\theta+4}{2}\right)$ $3=(\theta^2-2\theta+3)(-\theta^2-2\theta+1)$ $5=5$
111	$x^3+3x+8=0$	$-2^2.3^3.17$	$-3^3.17$	-459	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	$2\theta+3$	111 : $2=\left(\frac{\theta^2-\theta+4}{2}\right)(-\theta-1)$ $3=\left(\frac{\theta^2-\theta+6}{2}\right)^3(-2\theta-3)^*$ $5=5$
112	$x^3-3x+8=0$	$-2^2.3^4.5$	$-2^2.3^4.5$	-1620	3	......	$2\theta+5 \mp k^3$	112 $\begin{cases} (2)=(2,\theta-1)^2(2,\theta+2) \ (3)=(3,\theta-1)^3 \ (5)=(5,\theta+1)^2(5,\theta-2) \\ (7)=(7) \ (11)=(11,\theta+4)(11,\theta^2-4\theta+2) \end{cases}$
113	$x^3+4x+8=0$	$-2^6.31$	-31	-31	1	$\equiv 2.\frac{\theta^2}{4}, \frac{\theta}{2}, 1$	......	113 :
114	$x^3-4x+8=0$	$-2^6.23$	-23	-23	1	$\equiv 3.\frac{\theta^2}{4}, \frac{\theta}{2}, 1$	......	114 :
115	$x^3+5x+8=0$	$-2^2.557$	$-2^2.557$	-2228	1	......	......	115 $\begin{cases} 2=(-\theta-1)^2(4\theta^2-5\theta+26) \ 3=3 \ 5=(2\theta^2-3)(26\theta^2-32\theta+169) \\ 7=(\theta^2-\theta+7)(-\theta^2-\theta+1) \ 11=11 \ 13=(4\theta+5)(16\theta^2-20\theta+105) \end{cases}$
116	$x^3-5x+8=0$	$-2^2.307$	-307	-307	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	......	116 : $2=\left(\frac{\theta^2+\theta-4}{2}\right)(-\theta+1)$ $3=\left(\frac{-\theta^2-\theta+6}{2}\right)\left(\frac{\theta^2-\theta+2}{2}\right)$
117	$x^3+6x+8=0$	$-2^5.3^4$	$-2^3.3^4$	-648	3	$\frac{\theta^2}{2}, \theta, 1$	$\theta+1 \mp k^3$	117 $\begin{cases} (2)=\left(2,\frac{\theta^2+2\theta+2}{2}\right)^2\left(2,\frac{\theta^2}{2}\right) \ (3)=(3,\theta-1)^3 \\ (5)=(5,\theta-1)(5,\theta^2+\theta+2) \ (7)=(7,\theta-2)(7,\theta^2+2\theta+3) \end{cases}$
118	$x^3-6x+8=0$	$-2^5.3^3$	$-2^3.3^3$	-216	1	$\frac{\theta^2}{2}, \theta, 1$	$\theta+3$	118 : $2=\left(\frac{\theta^2+2\theta-2}{2}\right)^2\left(\frac{\theta^2-4\theta+4}{2}\right)$ $3=(-\theta+1)^3(\theta+3)^*$
119	$x^3+7x+8=0$	red.						119 :
120	$x^3-7x+8=0$	$-2^2.89$	$-2^2.89$	-356	1	......	......	120 : $2=(-\theta+1)^2(-2\theta^2-3\theta+10)$ $3=3$ $5=5$
121	$x^3+8x+8=0$	$-2^6.59$	-59	-59	1	$\equiv 9.\frac{\theta^2}{4}, \frac{\theta}{2}, 1$	......	121 :
122	$x^3-8x+8=0$	red.						122 :
123	$x^3+8x+1=0$	$-5^2.83$	-83	-83	1	$\frac{\theta^2+2\theta+2}{5}, \theta, 1$	θ	123 : $2=\left(\frac{\theta^2+2\theta+7}{5}\right)\left(\frac{\theta^2-3\theta+7}{5}\right)$
124	$x^3-8x+1=0$	43.47	43.47	2021	1	......	θ	124 $\begin{cases} 2=(\theta+3)(\theta^2-3\theta+1) \ 3=(-2\theta^2+6\theta-1)(2\theta^2+2\theta-11) \ 5=5 \\ 7=(\theta-2)(\theta^2+2\theta-4) \end{cases}$
125	$x^3+8x+2=0$	$-2^2.7^2.11$	$-2^2.11$	-44	1	$\frac{\theta^2-3\theta+3}{7}, \theta, 1$	......	125 :
126	$x^3-8x+2=0$	$2^2.5.97$	$2^2.5.97$	1940	1	......	$\theta^2-3\theta+1$	126 $\begin{cases} 2=(-\theta)^3(16\theta^2+4\theta-127)^* \ 3=(\theta^2-7)(\theta^2+2\theta-1) \\ 5=(\theta-1)^2(-\theta^2+9) \ 7=7 \end{cases}$

PART I.—Continued.

		$d\,(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
127	$x^3+3x+3=0$	$-29\,.\,79$	$-29\,.\,79$	-2291	1	......	......	127 $\begin{cases} 2 = (8\theta^2-3\theta+65)(\theta^2+3\theta+1) \;\; 3=(-\theta)(2\theta^2-2\theta-1)(103\theta^2-38\theta+838) \;\; 5=5 \\ 7 = (7\theta^2+8\theta+2)(442\theta^2-163\theta+3596) \;\; 11=(2\theta^2-\theta+16)(\theta^2+4\theta+2) \;\; 13=13 \end{cases}$
128	$x^3-3x+3=0$	red.						128 :
129	$x^3+8x+4=0$	$-2^4\,.\,5\,.\,31$	$-2^2\,.\,5\,.\,31$	-610	1	$\frac{\theta^2}{2}, \theta, 1$	$2\theta+1$	129 : $2 = \left(\frac{\theta^2}{2}\right)^3 (140\theta^2-68\theta+1153)^* \;\; 3=3 \;\; 5=\left(\frac{\theta^2+6}{2}\right)^2(\theta+1)$
130	$x^3-8x+4=0$	$2^4\,.\,101$	$2^2\,.\,101$	404	1	$\frac{\theta^2}{2}, \theta, 1$	$2\theta-1$	130 : $2 = \left(\frac{\theta^2}{2}\right)^3 (132\theta^2+68\theta-1023)^* \;\; 3=(\theta-1)(\theta^2+\theta-7)$
131	$x^3+8x+5=0$	$-7\,.\,389$	$-7\,.\,389$	-2723	1	......	......	131 $\begin{cases} 2 = (5\theta^2-3\theta+42)(-\theta^2+\theta+1) \;\; 3=3 \;\; 5=(-\theta)(\theta^2+8) \\ 7 = (-2\theta-1)^2(20\theta^2-12\theta+167) \;\; 11=11 \;\; 13=(2\theta^2-\theta+16)(\theta^2-4\theta^2-2) \end{cases}$
132	$x^3-8x+5=0$	1373	1373	1373	1	......	......	132 : $2 = (\theta-1)(\theta^2+\theta-7) \;\; 3=(\theta-2)(\theta^2+2\theta-4) \;\; 5=(-\theta)(\theta^2-8) \;\; 7=7$
133	$x^3+8x+6=0$	$-2^2.5.151$	$-2^2.5.151$	-3020	3	......	......	133 $\begin{cases} (2)=(2,\theta)^3 \;\; (3)=(\theta+1)(3,\theta)(3,\theta-1) \;\; (5)=(5,\theta-2)^2(5,\theta-1) \\ (7)=(7) \;\; (11)=(11) \;\; (13)=(13) \end{cases}$
134	$x^3-8x+6=0$	$2^2\,.\,269$	$2^2\,.\,269$	1076	1	......	$3\theta^2-10\theta+7$	134 $\begin{cases} 2 = (\theta-2)^3(7\theta^2+16\theta-19)^* \;\; 3=(-\theta-3)(\theta^2-3\theta+1) \;\; 5=5 \\ 7 = (-2\theta^2-4\theta+7)(-2\theta^2+4\theta+1) \end{cases}$
135	$x^3+8x+7=0$	-3371	-3371	-3371	1	......	......	135 $\begin{cases} 2 = (\theta+1)(\theta^2-\theta+9) \;\; 3=3 \;\; 5=5 \;\; 7=(-\theta)(\theta^2+8) \\ 11=(6\theta^2-5\theta+52)(\theta^2+8\theta+6) \;\; 13=13 \end{cases}$
136	$x^3-8x+7=0$	red.						136 :
137	$x^3+9=0$	-3^7	-3^5	-243	1	$\equiv 11$	......	137 :
138	$x^3+x+9=0$	$-7\,.\,313$	$-7\,.\,313$	-2191	2	......	$\theta-2\pm k^2$	138 $\begin{cases} (2)=(2) \;\; (3)=(3,\theta)(3,\theta^2+1) \;\; (5)=(5) \;\; (7)=(\theta+1)(7,\theta+3)^2 \\ (11)=(\theta-1)(11,\theta-4)(11,\theta+5) \;\; (13)=(13,\theta-3)(13,\theta^2+3\theta+10) \end{cases}$
139	$x^3-x+9=0$	$-37\,.\,59$	$-37\,.\,59$	-2183	2	......	$\theta^2-5\pm k^2$	139 $\begin{cases} (2)=(2) \;\; (3)=(-\theta-2)(3,\theta)(3,\theta+1) \;\; (5)=(5,\theta-2)(5,\theta^2+2\theta+3) \;\; (7)=(7) \\ (11)=(11,\theta-3)(11,\theta^2+3\theta+8) \;\; (13)=(13) \end{cases}$
140	$x^3+2x+9=0$	$-7\,.\,317$	$-7\,.\,317$	-2219	2	......	......	140 $\begin{cases} (2)=(2,\theta+1)(2,\theta^2+\theta+1) \;\; (3)=(\theta+2)(3,\theta)(3,\theta+1) \;\; (5)=(5) \\ (7)=(4\theta^2-7\theta+20)(7,\theta-2)^2 \;\; (11)=(11) \;\; (13)=(13) \end{cases}$
141	$x^3-2x+9=0$	$-5\,.\,433$	$-5\,.\,433$	-2155	1	......	$16\theta^2-25\theta-152$	141 $\begin{cases} 2 = (-\theta^2-2\theta+1)(3\theta^2-7\theta+11) \;\; 3=(20\theta^2-48\theta+75)(4\theta^2-23) \\ 5 = (-\theta-2)^2(2\theta^2-5\theta+8) \;\; 7=7 \;\; 11=11 \\ 13=(-\theta+2)(2\theta+5)(\theta^2-2\theta+4) \end{cases}$
142	$x^3+3x+9=0$	$-3^3.5.17$	$-3\,.\,5\,.\,17$	-255	1	$\frac{\theta^2}{3}, \theta, 1$	......	142 : $2=2 \;\; 3=\left(\frac{\theta^2+3}{3}\right)^2\left(\frac{\theta^2}{3}\right)$
143	$x^3-3x+9=0$	$-3^3.7.11$	$-3\,.\,7\,.\,11$	-231	1	$\frac{\theta^2}{3}, \theta, 1$	......	143 : $2=2 \;\; 3=\left(\frac{\theta^2-3}{3}\right)^2\left(\frac{\theta^2}{3}\right)$
144	$x^3+4x+9=0$	$-7\,.\,349$	$-7\,.\,349$	-2443	2	......	$\theta^2-4\theta-8$	144 $\begin{cases} (2)=(2,\theta+1)(2,\theta^2+\theta+1) \;\; (3)=(2\theta+3)(4\theta^2-6\theta+25) \\ (5)=(5,\theta-2)(5,\theta^2+2\theta+8) \;\; (7)=(\theta+2)(7,\theta-1)^2 \\ (11)=(11,\theta-5)(11,\theta^2+5\theta+29) \end{cases}$

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
145	$x^3-4x+9=0$	-1931	-1931	-1931	2	......	......	145 $\begin{cases}(2) = (2, \theta+1)(2, \theta^2+\theta+1) \ (3) = (-\theta^2-2\theta+2)(3, \theta)(3, \theta+1) \ (5) = (5) \\ (7) = (7) \ (11) = (11)\end{cases}$
146	$x^3+5x+9=0$	-2687	-2687	-2687	2	......	$3\theta+4 \mp k^2$	146 $\begin{cases}(2) = (2) \ (3) = (-\theta-1)(3, \theta)(3, \theta-1) \ (5) = (5, \theta-1)(5, \theta^2+\theta+1) \\ (7) = (7) \ (11) = (11, \theta+3)(11, \theta^2-3\theta+3) \ (13) = (13)\end{cases}$
147	$x^3-5x+9=0$	$-7 . 241$	$-7 . 241$	-1687	1	......	$7\theta^2-20\theta+22$	147 $\begin{cases}2 = 2 \ 3 = (\theta+3)(\theta^2-3\theta+4) \ 5 = (-\theta+1)(\theta^2+\theta-4) \\ 7 = (-\theta+2)^2(\theta^2+\theta-5) \ 11 = (-\theta-2)(\theta^2-2\theta-1)\end{cases}$
148	$x^3+6x+9=0$	$-3^3 . 113$	$-3 . 113$	-339	1	$\frac{\theta^2}{3}, \theta, 1$	......	148 : $2 = (-\theta-1)(\theta^2-\theta+7) \ 3 = \left(\frac{\theta^2+6}{3}\right)^2 \left(\frac{\theta^2}{3}\right) \ 5 = 5$
149	$x^3-6x+9=0$	red.						149 :
150	$x^3+7x+9=0$	-3559	-3559	-3559	2	......	$\theta+1 \mp k^2$	150 $\begin{cases}(2) = (2) \ (3) = (3, \theta)(3, \theta^2+1) \ (5) = (5) \ (7) = (7) \ (11) = (11) \\ (13) = (\theta+2)(13, \theta+3)(13, \theta-5)\end{cases}$
151	$x^3-7x+9=0$	$-5 . 163$	$-5 . 163$	-815	1	......	... ..	151 : $2 = 2 \ 3 = (-\theta+1)(-\theta+2)(-\theta-3) \ 5 = (-\theta^2-\theta+7)^2(3\theta^2-9\theta+8) \ 7 = 7$
152	$x^3+8x+9=0$	red.						152 :
153	$x^3-8x+9=0$	-139	-139	-139	1	... ..	$\theta-2$	153 : $2 = (-\theta+1)(\theta^2+\theta-7) \ 3 = (-2\theta^2-2\theta+15)(2\theta^2-6\theta+5)$
154	$x^3+9x+9=0$	$-3^6 . 7$	$-3^4 . 7$	-567	1	$\frac{\theta^2}{3}, \theta, 1$	$\theta+1$	154 : $2 = 2 \ 3 = \left(\frac{\theta^2}{3}\right)^3 (12\theta^2-11\theta+118)^* \ 5 = \left(\frac{\theta^2+6}{3}\right)\left(\frac{11\theta^2-3\theta+3}{3}\right)$
155	$x^3-9x+9=0$	3^6	3^4	81	1	......	......	155 :
156	$x^3+9x+1=0$	$-3^3 . 109$	$-3 . 109$	-327	1	$\frac{\theta^2-\theta+1}{3}, \theta, 1$	$\theta, 9\theta+1$	156 : $2 = 2 \ 3 = \left(\frac{\theta^2-\theta+1}{3}\right)^2 \left(\frac{7\theta^2-\theta+64}{3}\right) \ 5 = \left(\frac{\theta^2+5\theta+1}{3}\right)\left(\frac{11\theta^2-2\theta+98}{3}\right)$
157	$x^3-9x+1=0$	$3^3 . 107$	$3 . 107$	321	1	$\frac{\theta^2-\theta+1}{3}, \theta, 1$	$\theta, \theta+3, \theta-3$	157 : $2 = 2 \ 3 = \left(\frac{\theta^2-\theta+1}{3}\right)^2 (11\theta^2-\theta+100)$
158	$x^3+9x+2=0$	$-2^4 . 3^3 . 7$	$-2^2 . 3^3 . 7$	-756	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	$\frac{5\theta^2-17\theta-4}{2}$	158 $\begin{cases}2 = \left(\frac{-\theta^2-\theta}{2}\right)^2 \left(\frac{59\theta^2-13\theta+534}{2}\right) \ 3 = \left(\frac{\theta^2-\theta}{2}\right)^3 \left(\frac{267\theta^2-59\theta+4834}{2}\right)^* \\ 5 = 5 \ 7 = (5\theta^2-\theta+45)^2(2\theta^2+14\theta+3)\end{cases}$
159	$x^3-9x+2=0$	$2^3 . 3^3 . 13$	$2^3 . 3^3 . 13$	2808	1	......	......	159 $\begin{cases}2 = (-\theta-3)^2(7\theta^2-22\theta+6) \ 3 = (-2\theta^2+6\theta-1)^3(-30\theta^2-22\theta+221)^* \\ 5 = (\theta^2-3\theta+1)(-\theta^2+\theta+13) \ 7 = 7 \ 11 = 11\end{cases}$
160	$x^3+9x+3=0$	$-3^5 . 13$	$-3^5 . 13$	-3159	1	......	......	160 $\begin{cases}2 = 2 \ 3 = (-\theta)^3(9\theta^2-3\theta+82)^* \ 5 = 5 \ 7 = (\theta+1)(\theta^2-\theta+10) \\ 11 = 11 \ 13 = (3\theta^2-\theta+28)(-2\theta^2+\theta+1)\end{cases}$
161	$x^3-9x+3=0$	$3^5 . 11$	$3^5 . 11$	2673	1	......	$3\theta-1$	161 $\begin{cases}2 = 2 \ 3 = (-\theta)^3(9\theta^2+3\theta-80)^* \ 5 = (\theta-1)(\theta^2+\theta-8) \\ 7 = (\theta-2)(\theta^2+2\theta-5) \ 11 = (\theta^2+2\theta-4)^2(4\theta^2-13\theta+5)\end{cases}$
162	$x^3+9x+4=0$	$-2^2 . 3^3 . 31$	$-2^2 . 3^3 . 31$	-3348	1	......	......	162 $\begin{cases}2 = (16\theta^2-7\theta+147)^2(10\theta^2-37\theta-18) \\ 3 = (3\theta^2-\theta-1)^3(6244913\theta^2-2718291\theta+57387437)^* \\ 5 = (2\theta+1)(4\theta^2-2\theta+37) \ 7 = (-2\theta^2+6\theta+3)(78\theta^2-34\theta+717) \\ 11 = (1916\theta^2-834\theta+17607)(48\theta^2-14\theta-3) \\ 13 = (12477\theta^2-5431\theta+114657)(133\theta^2+51\theta-3)\end{cases}$
163	$x^3-9x+4=0$	$2^2 . 3^3 . 23$	$3^3 . 23$	621	1	$\frac{\theta^2+\theta}{2}, \theta, 1$	$-4\theta^2+26\theta-3$	163 : $2 = \left(\frac{\theta^2+\theta-8}{2}\right)(\theta-1) \ 3 = (2\theta-1)^3(488\theta^2+222\theta-4291)^* \ 5 = 5$

PART I.—Continued.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
164	$x^3+9x+5=0$	$-3^3.7.19$	$-3^3.7.19$	-3591	1	.....	$-4\theta^2+22\theta+13$	164 $\begin{cases} 2 = 2 \;\; 3=(-2\theta-1)^3(52\theta^2-28\theta+483)^* \;\; 5=(-\theta)(\theta+1)(2\theta^2-\theta+19) \\ 7 = (2\theta^2-\theta+18)^2(-\theta^2+5\theta+3) \;\; 11=(11\theta^2-6\theta+102)(3\theta^2+7\theta+3) \\ 13=13 \end{cases}$
165	$x^3-9x+5=0$	$3^3.83$	$3^3.83$	1241	1	......	$-\theta^2+4\theta-2$	165 : $2=2 \;\; 3=(\theta-1)^3(5\theta^2+3\theta-43)^* \;\; 5=(-\theta)(-\theta+3)(-\theta-3) \;\; 7=7$
166	$x^3+9x+6=0$	$-2^4.3^5$	-3^5	-243	1	$\equiv 11$	......	166 :
167	$x^3-9x+6=0$	$2^3.3^5$	$2^3.3^5$	1944	1	......	......	167 $\begin{cases} 2 = (\theta-1)^2(-3\theta^2-2\theta+26) \;\; 3=(\theta^2-5\theta+3)^3(14007\theta^2+9885\theta-119087)^* \\ 5 = 5 \;\; 7=(-\theta^2-\theta+7)(-\theta^2+\theta+1) \end{cases}$
168	$x^3+9x+7=0$	$-3^3.157$	$-3^3.157$	-4239	1	......	$\theta^2-2\theta-2$	168 $\begin{cases} 2 = 2 \;\; 3=(\theta+1)^3(15\theta^2-11\theta+143)^* \;\; 5=5 \;\; 7=(-\theta)(\theta^2+9) \\ 11=(\theta^2-\theta+9)(\theta^2+2\theta+2) \;\; 13=13 \;\; 17=(-\theta+1)(\theta^2+\theta+10) \end{cases}$
169	$x^3-9x+7=0$	$3^3.59$	$3^3.59$	1593	1	......	$\theta-1$	169 $\begin{cases} 2 = 2 \;\; 3=(\theta-2)^3(3\theta^2+7\theta-10)^* \;\; 5=(-\theta^2-3\theta+4)(-\theta^2+\theta+3) \\ 7 = (-\theta)(-\theta+3)(-\theta-3) \end{cases}$
170	$x^3+9x+8=0$	$-2^2.3^3.43$	$-2^2.3.43$	-516	1	$\frac{\theta^2+\theta+1}{3}, \theta, 1$	......	170 : $2=(\theta+1)^2(6\theta^2-5\theta+58) \;\; 3=\left(\frac{\theta^2+\theta+1}{3}\right)^2\left(\frac{10\theta^2-8\theta+97}{3}\right) \;\; 5=5$
171	$x^3-9x+8=0$	red.						171 :
172	$x^3+10=0$	$-2^2.3^3.5^2$	$-2^2.3.5^2$	-300	1	......	$-3\theta^2-6\theta+1$	172 : $2=(-\theta-2)^3(-3\theta^2-6\theta+1)^* \;\; 3=\left(\frac{\theta^2-\theta+1}{3}\right)^2\left(\frac{\theta^2+2\theta+1}{3}\right)$

PART II.

		$d(\theta)$	Δ		h	Basis.	Units.	Factorization of Rational Primes.
1	$x^3-x^2+x+1=0$	$-2^2 . 11$	$-2^2 . 11$	-44	1	......	......	1 :
2	$x^3+x^2+x+2=0$	-83	-83	-83	1	......	$\theta+1$	$2 : 2=(-\theta)(\theta^2+\theta+1)$
3	$x^3-x^2+x+2=0$	-139	-139	-139	1	......	$\theta+1$	$3 : 2=(-\theta)(\theta^2-\theta+1)\ \ 3=(-\theta+1)(\theta^2+1)$
4	$x^3-x^2-x+2=0$	-59	-59	-59	1	......	$\theta-1,\ \theta+1$	$4 : 2=(-\theta)(\theta^2-\theta-1)$
5	$x^3+x^2+2x+1=0$	-23	-23	-23	1	......	......	5 :
6	$x^3-x^2+2x+1=0$	$-3 . 29$	$-3 . 29$	-87	1	......	θ	$6 : 2=2$
7	$x^3+x^2-2x+1=0$	-31	-31	-31	1	......	......	7 :
8	$x^3-x^2-2x+1=0$	7^2	7^2	49	1	......	......	8 :
9	$x^3-x^2+2x+2=0$	$-2^3 . 5^2$	$-2^3 . 5^2$	-200	1	......	......	$9 : 2=(-\theta)^2(\theta^2-2\theta+3)\ \ 3=3$
10	$x^3+x^2-2x+2=0$	$-2^3 . 19$	$-2^3 . 19$	-152	1	......	$\theta^2-\theta+1$	$10 : 2=(-\theta)^2(-\theta^2-2\theta+1)\ \ 3=3$
11	$x^3-2x^2+x+2=0$	$-2^2 . 29$	$-2^2 . 29$	-116	1	......	......	$11 : 2=(\theta+1)^2(-\theta)\ \ 3=3$
12	$2x^3-x^2+x+2=0$	-503	-503	-503	1	$\frac{\theta^2+\theta}{2},\ \theta,\ 1$	......	$12 : 2=(2\theta^2-5\theta+11)(\theta^2-2)\ \ 3=3\ \ 5=(2\theta+3)(4\theta^2-10\theta+23)$
	$x^3-x^2+2x+8=0$	$-2^2 . 503$						
13	$x^3-2x^2+2x+2=0$	$-2^2 . 51$	$-2^2 . 51$	-204	1	......	$-\theta^2-5\theta+7$	$13 : 2=(-\theta)^3(-\theta^2+5\theta-7)^*\ \ 3=(-\theta+1)(\theta+1)(\theta^2-2\theta+3)$
14	$x^3+2x^2-2x+2=0$	$-2^2 . 67$	$-2^2 . 67$	-268	1	......	$\theta+3$	$14 : 2=(-\theta)^3(\theta+3)^*\ \ 3=(-\theta+1)(\theta^2+3\theta+1)$
15	$x^3-2x^2-2x+2=0$	$2^2 . 37$	$2^2 . 37$	148	1	......	$\theta+1,\ \theta-1$	$15 : 2=(-\theta)^3(2\theta^2-3\theta-5)^*$
16	$2x^3+x^2+2x+2=0$	$-2^2 . 89$	$-2^2 . 89$	-356	1	$\frac{\theta^2+\theta}{2},\ \theta,\ 1$	.....	$16 : 2=\left(\frac{\theta^2-\theta+4}{2}\right)^2\left(\frac{-\theta^2-3\theta-2}{2}\right)\ \ 3=3\ \ 5=5$
	$x^3+x^2+4x+8=0$	$-2^4 . 89$						
17	$2x^3-x^2+2x+2=0$	$-2^2 . 157$	$-2^2 . 157$	-628	1	$\frac{\theta^2+\theta}{2},\ \theta,\ 1$	......	$17 \begin{cases} 2=\left(\frac{\theta^2+\theta}{2}\right)^2\left(\frac{25\theta^2-55\theta+166}{2}\right)\ \ 3=(5\theta^2-11\theta+33)(\theta^2-3\theta-5) \\ 5=\left(\frac{\theta^2-\theta+2}{2}\right)\left(\frac{3\theta^2-7\theta+18}{2}\right)\ \ 7=7 \end{cases}$
	$x^3-x^2+4x+8=0$	$-2^4 . 157$						
18	$2x^3+x^2-2x+2=0$	$-2^2 . 3 . 43$	$-2^2 . 3 . 43$	-516	1	$\frac{\theta^2+\theta}{2},\ \theta,\ 1$	......	$18 \begin{cases} 2=\left(\frac{\theta^2+3\theta}{2}\right)^2\left(\frac{7\theta^2-15\theta+18}{2}\right)\ \ 3=(9\theta^2-19\theta+23)^2(-6\theta^2+26\theta+139) \\ 5=5 \end{cases}$
	$x^3+x^2-4x+8=0$	$-2^4 . 3 . 43$						
19	$2x^3-x^2-2x+2=0$	$-2^2 . 53$	$-2^2 . 53$	-212	1	$\frac{\theta^2+\theta}{2},\ \theta,\ 1$	$\theta^2+\theta-3$	$19 : 2=\left(\frac{-\theta^2-\theta+4}{2}\right)^2\left(\frac{\theta^2-3\theta+2}{2}\right)\ \ 3=3$
	$x^3-x^2-4x+8=0$	$-2^4 . 53$						

On Certain Properties of the Plane Cubic Curve in Relation to the Circular Points at Infinity.

By R. A. Roberts.

Part I.—*On some Methods of Generating the Plane Cubic Curve.*

I propose to investigate here some methods of generating a plane cubic curve. I begin by obtaining the cubic as a locus of a point P as follows: If perpendiculars be drawn from P on the sides of a given triangle, then the circle passing through the feet of these perpendiculars cuts orthogonally a fixed circle. A triangle involves six constants and a circle three, so that we have nine constants, which is the number involved in the equation of the general cubic.

I observe that if we describe the conic with P as a focus and touching the sides of the triangle, then the circle passing through the feet of the perpendiculars from P on the sides of the triangle is the circle described on the transverse axis of the conic as diameter. Using trilinear coordinates and expressing that the product of the perpendiculars from the foci on a tangent is constant, we get the tangential equation of the conic in the form

$$(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma') \\ - B^2(\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C) = 0. \quad (1)$$

Now, if this conic touch the sides of the triangle, the coefficients of λ^2, μ^2, ν^2 must vanish. We thus have $\alpha\alpha' = \beta\beta' = \gamma\gamma' = B^2$. Hence, the tangential equation of the conic touching the lines α, β, γ and having the points α, β, γ as a focus, is

$$\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A)\mu\nu + \beta(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B)\nu\lambda \\ + \gamma(\alpha^2 + \beta^2 + 2\alpha\beta\cos C)\lambda\mu = 0. \quad (2)$$

I now obtain the equation of the director circle of this conic. The coordinates λ, μ, ν are proportional to $a\varpi_1$, $b\varpi_2$, $c\varpi_3$ respectively, where a, b, c are the sides of the triangle and ϖ_1, ϖ_2, ϖ_3 are the perpendiculars from the vertices of the

triangle on a line. Making this substitution, and putting $\varpi_1 = p - x_1 \cos\omega - y_1 \sin\omega$, $\varpi_2 =$ etc., and then putting $p = x\cos\omega + y\sin\omega$ for the tangents drawn to the conic (2) from x, y, we get an equation determining the directions of those tangents. Putting the sum of the coefficients of $\cos^2\omega$ and $\sin^2\omega$ equal to nothing, we get the condition that the tangents drawn to the conic from x, y should be at right angles to each other; that is, the condition that x, y should lie on the director circle of the conic. We have, then, for the equation of the director circle,

$$S = bc\alpha\,(\beta^2 + \gamma^2 + 2\beta\gamma\cos A)\,S_1 + ca\beta\,(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B)\,S_2 \\ + ab\gamma\,(\alpha^2 + \beta^2 + 2\alpha\beta\cos C)\,S_3 = 0, \quad (3)$$

where S_1, S_2, S_3 are the circles described on the three sides as diameters respectively, viz.: $S_1 = (x - x_2)(x - x_3) + (y - y_2)(y - y_3)$, etc.

Taking the origin of the Cartesian coordinates at the centre of the conic, we may write

$$S = (a\alpha + b\beta + c\gamma)(a\beta\gamma + b\gamma\alpha + c\alpha\beta)(x^2 + y^2 - A^2 - B^2),$$

where A, B are the principal semiaxes of the conic; that is,

$$S = 2\Delta\Sigma\,(x^2 + y^2 - A^2 - B^2), \quad (4)$$

where Δ is the area of the triangle and $\Sigma = a\beta\gamma + b\gamma\alpha + c\alpha\beta$, so that $\Sigma = 0$ is the equation of the circumscribing circle. Again, let one focus satisfy the equation $a\alpha + b\beta + c\gamma = 2\Delta$, then, substituting $\dfrac{B^2}{\alpha}$, $\dfrac{B^2}{\beta}$, $\dfrac{B^2}{\gamma}$ for α, β, γ respectively, we have

$$B^2\Sigma = 2\Delta\alpha\beta\gamma. \quad (5)$$

Hence, from (4) we get

$$S + 4\Delta^2\alpha\beta\gamma = 2\Delta\Sigma\,(x^2 + y^2 - A^2). \quad (6)$$

But $x^2 + y^2 - A^2 = 0$ is the circle described on the transverse axis of the conic as diameter; that is, the circle passing through the feet of the perpendiculars from α, β, γ on the sides of the triangle. Now, if this circle cut orthogonally a fixed circle, its coefficients are connected by a linear relation. Hence, expressing that the coefficients of $S + 4\Delta^2\alpha\beta\gamma$ are connected by a linear relation, we have

$$L\alpha\,(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) + M\beta\,(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B) \\ + N\gamma\,(\alpha^2 + \beta^2 + 2\alpha\beta\cos C) + 2P\alpha\beta\gamma = 0, \quad (7)$$

where L, M, N, P are constants determining the position of the fixed circle. It may be observed that

$$L = bc\,(S_1 - k^2), \quad M = ca\,(S_2 - k^2), \quad N = ab\,(S_3 - k^2), \quad P = 2\Delta^2, \tag{8}$$

where k is the radius of the fixed circle, and S_1, S_2, S_3 are the squares of the tangents drawn from the centre of the fixed circle to the circles described on the sides of the triangles as diameters respectively. Now, (7) represents a cubic passing through the vertices of the triangle, so that the points where the curve meets the sides again lie on a line, viz.:

$$MN\alpha + NL\beta + LM\gamma = \delta = 0. \tag{9}$$

The curve (7), then, can be written in the form

$$\delta\,(MN\beta\gamma + NL\gamma\alpha + LM\alpha\beta) + \{2LMN(L\cos A + M\cos B + N\cos C + P) \\ - M^2N^2 - N^2L^2 - L^2M^2\}\,\alpha\beta\gamma = 0, \tag{10}$$

so that, dividing by $\alpha\beta\gamma\delta$, the curve is of the form

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} + \frac{p}{\delta} = 0, \tag{11}$$

where l, m, n, p are constants. Now, I observe that the Hessian of the cubic

$$A_1 x_1^3 + A_2 x_2^3 + A_3 x_3^3 + A_4 x_4^3 = 0, \tag{12}$$

where A_1, A_2, A_3, A_4 are constants and $x_1 + x_2 + x_3 + x_4 = 0$ identically is

$$\frac{1}{A_1 x_1} + \frac{1}{A_2 x_2} + \frac{1}{A_3 x_3} + \frac{1}{A_4 x_4} = 0, \tag{12}$$

and that corresponding points on the latter cubic are connected by the relations

$$A_1 x_1 x_1' = A_2 x_2 x_2' = A_3 x_3 x_3' = A_4 x_4 x_4'. \tag{13}$$

We see thus that points on the cubic (7) such that $\alpha\alpha' = \beta\beta' = \gamma\gamma'$, viz., foci of the conic (2) are corresponding points on the curve, namely, points such that the tangents thereat intersect on the cubic. Now, the cubic (7) is the Hessian of a cubic of the form

$$MN\alpha^3 + NL\beta^3 + LM\gamma^3 + \theta\delta^3 = 0, \tag{14}$$

so that the polar line, that is, the pole of one point with regard to the polar conic of the other, of the circular points, for which $\alpha\alpha' = \beta\beta' = \gamma\gamma'$ is $\delta = 0$, viz., the line passing through the three points on the curve corresponding to the vertices of the triangle. Now, there are three cubics of which a given cubic is the

Hessian, corresponding to the three systems of corresponding points, so that the cubic can be written in the form (7) in three ways. Thus the cubic can be generated in the manner described in three ways.

A general circular cubic cannot be generated in this manner. For, if the cubic (7) is circular, $P=0$; but then the circular points are corresponding points on the curve; that is, the double focus of the curve is on itself. In such a case the circle passing through the feet of the perpendiculars has its centre on a fixed line instead of cutting a fixed circle orthogonally, as is evident from the fact that the conic (2) then touches another fixed line. The cubic is then the locus of a point P such that the feet of the perpendiculars from P on the sides of a quadrilateral lie on a circle.

If the fixed circle satisfy a certain relation, the locus breaks up into a conic passing through the vertices of the triangle and a right line. This relation is, from (10),

$$2LMNP = M^2N^2 + N^2L^2 + L^2M^2 - 2LMN(L\cos A + M\cos B + N\cos C);$$

that is, from (8),

$$\begin{aligned} 4\Delta^2(S_1-k^2)(S_2-k^2)(S_3-k^2) = {} & a^2(S_2-k^2)(S_3-k^2)(S_1-S_2)(S_1-S_3) \\ & + b^2(S_3-k^2)(S_1-k^2)(S_2-S_3)(S_2-S_1) \\ & + c^2(S_1-k^2)(S_2-k^2)(S_3-S_1)(S_3-S_2). \qquad (15) \end{aligned}$$

Hence, selecting any given point as centre, we have a cubic for k^2, so that three circles satisfying the condition are determined. Again, I observe that if the cubic (7) has a node, that point is situated at the centre of one of the circles touching the sides of the triangle; for instance, for the centre of the inscribed circle the cubic is

$$L\alpha(\beta-\gamma)^2 + M\beta(\gamma-\alpha)^2 + N\gamma(\alpha-\beta)^2 = 0, \qquad (16)$$

so that, from (8), the fixed circle then satisfies the condition

$$\begin{aligned} \tfrac{1}{4}(b+c-a)(c+a-b)(a+b-c) + (b+c-a)S_1 & \\ + (c+a-b)S_2 + (a+b-c)S_3 - k^2(a+b+c) = 0, & \qquad (17) \end{aligned}$$

that is, it must cut orthogonally the inscribed circle, as it can easily be proved that the result of putting $k=0$ in (17) gives the equation of the inscribed circle.

I now proceed to show that the same cubic can be generated if the circle

$$x^2 + y^2 = A^2 + B^2 - mB^2 \tag{18}$$

cut orthogonally a fixed circle.

We have from (4),

$$S = 2\Delta\Sigma\,(x^2 + y^2 - A^2 - B^2),$$

where, from (3), $S = bca\,(\beta^2 + \gamma^2 + 2\beta\gamma \cos A) + ca\beta\,(\gamma^2 + \alpha^2 + 2\gamma\alpha \cos B)$
$+ ab\gamma\,(\alpha^2 + \beta^2 + 2\alpha\beta \cos C)$;

also from (5), $B^2\Sigma = 2\Delta\alpha\beta\gamma$, where $\Sigma = a\beta\gamma + b\gamma\alpha + c\alpha\beta$. Hence,

$$S + 4m\Delta^2\alpha\beta\gamma = 2\Delta\Sigma\,\{x^2 + y^2 - A^2 - B^2 + mB^2), \tag{19}$$

so that to express that (18) cuts a fixed circle orthogonally, we write down a linear relation connecting the coefficients of the circle $S + 4m\Delta^2\alpha\beta\gamma = 0$, when we obtain a cubic of the form (7), where

$$L = bc\,(S_1 - k^2),\quad M = ca\,(S_2 - k^2),\quad N = ab\,(S_3 - k^2),\quad P = 2m\Delta^2. \tag{20}$$

From these equations it is easy to see that, if m be considered indeterminate, the fixed circle is not necessarily given, but may be replaced by any other circle passing through two given points, its centre lying on the line

$$La\,(S_2 - S_3 + Mb\,(S_3 - S_1) + Nc\,(S_1 - S_2) = 0. \tag{21}$$

Hence, making the radius vanish, we have from (20),

$$\frac{S_1}{La} = \frac{S_2}{Mb} = \frac{S_3}{Nc}, \tag{22}$$

which determines two points mutually inverse with regard to the polar circle of the triangle, as the latter circle is the Jacobian circle of S_1, S_2, S_3. We see thus that two circles of the system (18) can be determined, that is, two constant values of m can be assigned, so that the circle (19) passes through a fixed point. Again, if a certain condition be satisfied by the fixed circle similar to (15), the cubic breaks up into a line and a conic. This condition, corresponding to a given line, supplies a relation connecting m with the fixed circle.

Suppose $m = 2$, then the circle (18) is $x^2 + y^2 = A^2 - B^2$, that is, it is the circle described on the line joining the foci of the conic as diameter. Expressing then that $S + 8\Delta^2\alpha\beta\gamma$ cuts a fixed circle orthogonally, we obtain a general cubic of the form (7). Now, the foci of the conic are corresponding points on the cubic, so that we see that the circle described on the line joining a pair of cor-

responding points as diameter cuts orthogonally a fixed circle. This can be proved otherwise thus: A pair of corresponding points on the curve are conjugate with regard to all the polar conics of the cubic of which the given curve is the Hessian. Now, the equation of the polar conics being of the form $lU_1 + mU_2 + nU_3 = 0$, one of the system is a circle. But when two points are conjugate with regard to a circle, U say, the circle described on the line joining them as diameter cuts U orthogonally. In the particular case, it may be observed, when the Cayleyan has a focus on the cubic, at P say, that is, when U breaks up into factors, all the pairs of corresponding points subtend right angles at P.

I now proceed to find the locus of the centre of the conic (2), when the circle described on the transverse axis as diameter cuts orthogonally a fixed circle, or, in other words, the locus of the middle point of pairs of corresponding points on a given cubic. From the first point of view, as I shall show, this may be considered as another method of generating a general cubic. Let a, b be half the principal axes of the conic, and α, β, γ the perpendiculars from the centre on the sides of the triangle. Then

$$\alpha^2 = a^2 \cos^2(\theta - a) + b^2 \sin^2(\theta - a), \tag{23}$$

and similar values for β, γ, where a, β, γ, θ are the angles which the perpendiculars and the axis major make with a fixed line respectively. Eliminating θ and b, we obtain

$$\sin A \sqrt{(a^2 - \alpha^2)} + \sin B \sqrt{(a^2 - \beta^2)} + \sin C \sqrt{(a^2 - \gamma^2)} = 0, \tag{24}$$

where A, B, C are the angles of the triangle. Now, if the circle described on the axis major as diameter cuts orthogonally a circle $S = x^2 + y^2 + \text{etc.}$, we have $a^2 = S$, so that (24) becomes

$$\sin A \sqrt{(S - \alpha^2)} + \sin B \sqrt{(S - \beta^2)} + \sin C \sqrt{(S - \gamma^2)} = 0, \tag{25}$$

which, at first sight, appears to represent a quartic, but, being divisible by the line at infinity, reduces to a cubic, as $S - \alpha^2$, $S - \beta^2$, $S - \gamma^2$ are easily seen to be parabolæ. The cubic (25) is, in fact, the envelope of the conic

$$l(S - \alpha^2) + m(S - \beta^2) + n(S - \gamma^2) = 0, \tag{26}$$

subject to the condition that it is a parabola, viz.:

$$mn \sin^2 A + nl \sin^2 B + lm \sin^2 C = 0. \tag{27}$$

We can obtain the equation of this cubic in another form, thus: Eliminating θ and a from (23), we get

$$\sin A\sqrt{(\alpha^2 - b^2)} + \sin B\sqrt{(\beta^2 - b^2)} + \sin C\sqrt{(\gamma^2 - b^2)} = 0, \tag{27}$$

then, observing that, since the director circle of the conic cuts orthogonally the polar circle P of the triangle, we have $a^2 + b^2 = P$, so that if $S - P = \delta$, namely, the radical axis of S and P, we get

$$\sin A\sqrt{(\alpha^2 + \delta)} + \sin B\sqrt{(\beta^2 + \delta)} + \sin C\sqrt{(\gamma^2 + \delta)} = 0, \tag{28}$$

which, being divisible by the line at infinity, represents a cubic. This curve, it may be observed, passes through the points where the lines joining the middle points of the sides meet the circle S and the radical axis of S and P.

Projecting, we find that the polar of a fixed line with regard to A, B is a cubic, where A, B are corresponding points on a given cubic.

In the case of the circular cubic, we may find the locus of the middle point of corresponding points thus: Let the curve be

$$lv\,(u^2 + c^2) + mu\,(v^2 + c^2) + n\,(u^2 + v^2) + 2puv = 0, \tag{29}$$

where u, v are circular coordinates, then corresponding points are such that $uu' = vv' = c^2$. Hence, for the middle point of corresponding points we may write

$$2u = u' + \frac{c^2}{u'}, \quad 2v = v' + \frac{c^2}{v'}, \tag{30}$$

where u', v' lie on (29). Hence, we obtain

$$\left(lu + mv + \frac{n}{c^2}\,uv + p\right)^2 = \frac{n^2}{c^4}\,(u^2 - c^2)(v^2 - c^2), \tag{31}$$

which represents a circular cubic with the points $u^2 = v^2 = c^2$ as foci. Thus the locus (31) is the transformation of the given curve (29) by a substitution (30) in which angles are preserved. The substitution is, in fact, a transformation from polar to elliptic coordinates, viz.:

$$r = \mu + \sqrt{(\mu^2 - c^2)}, \quad c\cos\theta = \nu, \tag{32}$$

where $$cx = \mu\nu, \quad cy = \sqrt{\{(\mu^2 - c^2)(c^2 - \nu^2)\}}, \tag{33}$$

and r, θ are polar coordinates. It may be observed that the anharmonic function of the locus can be expressed in terms of the similar function for the given

curve. For it is easy to see that the values of u corresponding to the foci of (29) are given by an equation of the form

$$(u^2+c^2-2\alpha u)(u^2+c^2-2\beta u)=0, \tag{34}$$

so that the foci of (31) are given by

$$(u-\alpha)(u-\beta)(u^2-c^2)=0, \tag{35}$$

but the anharmonic functions of both these biquadratics are expressible in terms of $\dfrac{c(\alpha-\beta)}{c^2-\alpha\beta}$.

If $n=0$ in (29), the double focus is at the origin on the curve, and the locus (31) reduces to the right line

$$lu+mv+p=0, \tag{36}$$

as we have seen already otherwise. That is, when the cubic is not circular, the locus reduces to a right line, if two asymptotes intersect on the curve.

I now proceed to show what geometrical relations are satisfied when a focus of the conic (2) lies on a general cubic passing through the vertices of the triangle of reference. Let Δ be the area of the triangle formed by the foci P, P' of the conic and a fixed point α', β', γ'; then Δ is proportional to

$$\begin{vmatrix} \dfrac{b^2}{\alpha}, & \dfrac{b^2}{\beta}, & \dfrac{b^2}{\gamma} \\ \alpha\ , & \beta\ , & \gamma \\ \alpha'\ , & \beta'\ , & \gamma' \end{vmatrix} = \frac{b^2}{\alpha\beta\gamma}V,$$

where

$$V=\alpha'\alpha(\beta^2-\gamma^2)+\beta'\beta(\gamma^2-\alpha^2)+\gamma'\gamma(\alpha^2-\beta^2). \tag{37}$$

Again, let t be the length of the tangent drawn from a fixed point to the circle passing through the feet of perpendiculars from P on the sides of the triangle of reference, then, if we have

$$t^2-k^2=\lambda\Delta, \tag{38}$$

where k and λ are constants, we see from (7) and (37) that the locus of P is

$$L\alpha(\beta^2+\gamma^2+2\beta\gamma\cos A)+M\beta(\gamma^2+\alpha^2+2\gamma\alpha\cos B)$$
$$N\gamma(\alpha^2+\beta^2+2\alpha\beta\cos C)+2P\alpha\beta\gamma+\lambda' V=0, \tag{39}$$

and this obviously represents a cubic circumscribing the triangle and satisfying no further relation with it, for we have six constants at our disposal,

namely, k, λ and the four depending upon the two fixed points. The locus of the other focus is evidently obtained by changing the sign of V in (39), as V becomes $-\dfrac{V}{\alpha^2\beta^2\gamma^2}$ when we substitute for α, β, γ their reciprocals. This mode of generation of the cubic holds also if we substitute the circle described on PP' as diameter for the circle passing through the feet of the perpendiculars from P; in fact, we might substitute any circle of the system (19).

If we want to generate a circular cubic we should take $P = 0$ in (39) and make V pass through the circular points. The first condition makes the circle the director circle of the conic (2), and the second requires that the point α', β', γ' should be at infinity. Thus the relation (38) becomes $t^2 - k^2 = \lambda\delta$, (40), where t is the length of the tangent drawn from a fixed point to the director circle of the conic and δ is the projection of PP' on a fixed line. It may be observed that k may be made to vanish in (40) without loss of generality. A general circular cubic is also generated when the circle described on PP' as diameter cuts a given line at an angle whose cosine is proportional to the cosine of the angle between PP' and a fixed direction. Similarly, a general cubic is generated when the circle on PP' as diameter meets a given circle at an angle ϕ, so that $\cos\phi$ is proportional to the perpendicular from a fixed point on PP'.

It may be worth while considering the case in which the two fixed points involved in (38) coincide for the circle described on PP' as diameter. It is easy to see then that the cubic is the locus of the focus P of a conic touching the sides of a triangle, subject to the condition that the circle described through the foci P, P' so as to contain a given angle, cuts orthogonally a fixed circle. The equation of the locus is found to be

$$\begin{aligned}\frac{\alpha'\alpha}{\sin A}\{(\beta^2+\gamma^2)\cos A + 2\beta\gamma\} &+ \frac{\beta'\beta}{\sin B}\{(\gamma^2+\alpha^2)\cos B + 2\gamma\alpha\} \\ &+ \frac{\gamma'\gamma}{\sin C}\{(\alpha^2+\beta^2)\cos C + 2\alpha\beta\} \\ &- \frac{(S'-k^2)}{2R\sin A\sin B\sin C}(\alpha\sin A + B\sin B + \gamma\sin C) \\ &\qquad\qquad \times(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C) \\ &+ m\{\alpha'\alpha(\beta^2-\gamma^2) + \beta'\beta(\gamma^2-\alpha^2)\gamma'\gamma(\alpha^2-\beta^2)\} = 0, \quad (41)\end{aligned}$$

where α', β', γ' are the coordinates of the centre of the fixed circle, S' is the square of the tangent drawn from its centre to the circumscribing circle, and k is

its radius, while m is the cotangent of the given angle. This form (41) contains ten constants, so that a cubic can be generated in this manner in a singly infinite number of ways. If $k = 0$, the foci P, P' subtend a constant angle at a fixed point, and the equation (41) then involving nine constants, the cubic can be generated in this manner in a finite number of ways.

I observe that a circular cubic can be written in the form (29) in three ways, the origin in each case being one of the points corresponding to the real point at infinity. In connection with this form, I proceed to investigate a mode of generation of the curve. Taking rectangular Cartesian coordinates and writing a conic referred to its principal axes in the form

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0. \tag{42}$$

the equation of the circle passing through x', y' and the points of contact of the tangents drawn from x', y' to $S = 0$ is

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)(x^2 + y^2) - \frac{xx'}{a^2}(x'^2 + y'^2 + c^2) - \frac{yy'}{b^2}(x'^2 + y'^2 - c^2) + c^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) = 0, \tag{43}$$

where $c^2 = a^2 - b^2$.

Hence, if this circle cut orthogonally the fixed circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0, \tag{44}$$

the locus of x', y' is

$$U = \frac{\alpha x}{a^2}(x^2 + y^2 + c^2) + \frac{\beta y}{b^2}(x^2 + y^2 - c^2) - (k^2 + c^2)\frac{x^2}{a^2} - (k^2 - c^2)\frac{y^2}{b^2} = 0, \tag{45}$$

which, it is easy to see, is of the same form as (29). There is no loss of generality in making the radius of (44) vanish, in which case the circle (43) passes through a fixed point. In that case, the cubic is the locus of the six vertices of a quadrilateral circumscribed about the conic, the lines drawn from the fixed point (44) to the four points of contact all making the same angle with the conic.

If the polar circle of the triangle formed by the tangents from P and their chord of contact cuts orthogonally a fixed circle, the locus of P is a cubic with

two rectangular asymptotes. For the equation of the polar circle is

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)(x^2+y^2)-\frac{2x'}{a^2}\left(a^2+b^2+\frac{c^2y'^2}{b^2}\right)x$$
$$-\frac{2y'}{b^2}\left(a^2+b^2-\frac{c^2x'^2}{a^2}\right)y+a^2+b^2+\frac{b^2}{a^2}x'^2+\frac{a^2}{b^2}y'^2=0, \quad (46)$$

and if this cuts the circle (44) orthogonally, the locus of x', y' is

$$V=\frac{2c^2}{a^2b^2}xy(\beta x-\alpha y)+(b^2+k^2)\frac{x^2}{a^2}+(a^2+k^2)\frac{y^2}{b^2}$$
$$-2(a^2+b^2)\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}\right)+a^2+b^2=0. \quad (47)$$

We can now generate the general cubic by means of these results, (45) and (47). Let S denote the circle (43) circumscribing the triangle formed by the tangents from P and their chord of contact, and let P denote the polar circle (46) of the same triangle, then let $\Sigma = mS+nP=0$; that is, let Σ be a circle coaxial with S and P and with its centre dividing in a constant ratio the line joining the centres of S and P. Now let Σ cut the given circle (44) orthogonally, then the locus of $P(x', y')$ is the general cubic $mU+nV=0$.

I now proceed to consider another method of generating a cubic curve. Let us consider a circle passing through points on the sides of a triangle so that the lines joining them to the opposite vertices form two sets of three concurrent lines, which, it is to be observed, is only equivalent to a single condition. Using areal coordinates, the equation

$$(x+y+z)(ll'x+mm'y+nn'z)=k(a^2yz+b^2zx+c^2xy) \quad (48)$$

represents a circle. Making this identical with

$$ll'x^2+mm'y^2+nn'z^2-(mn'+m'n)yz-(nl'+n'l)zx-(lm'+l'm)xy=0, \quad (49)$$

where the lines joining the vertices to the intersections with the opposite sides are $lx-my=0$, $l'x-m'y$, etc., intersecting in the points mn, nl, lm; $m'n'$, $n'l'$, $l'm'$ respectively, say these are P, P', so that l, m, n; l', m', n' are the coordinates of points inverse to P, P' with regard to the triangle of reference, we obtain

$$(m+n)(m'+n')=ka^2, \quad (n+l)(n'+l')=kb^2, \quad (l+m)(l'+m')=kc^2, \quad (50)$$

so that (48) becomes

$$(x+y+z)\{a^2(l+m)(l+n)A+b^2(m+l)(m+n)B+c^2(n+l)(n+m)C\}$$
$$=2(m+n)(n+l)(l+m)(a^2yz+b^2zx+c^2xy), \quad (51)$$

where $\quad A=my+nz-lx, \quad B=lx+nz-my, \quad C=lx+my-nz.$

Now, expressing that this circle (51) cuts orthogonally a given circle, we get a linear solution connecting the coefficients. We thus have a relation of the third degree connecting l, m, n, so that the inverse points of P, P' with regard to the triangle lie on a cubic circumscribing the triangle $l+m=0$, $l+n=0$, $m+n=0$, so that the points where it meets the sides again lie on a line. And it may be observed that the two points are corresponding points of the cubic. Now the triangle of reference involves six constants and the given circle involves three, so that we have the nine constants involved in a general cubic.

I now consider Grassman's method of generating a cubic. Let us consider two triangles whose vertices are A, B, C, A', B', C' respectively; then, if the lines joining P to A, B, C meet $B'C'$, $C'A'$, $A'B'$ in three collinear points, then the locus of P is the cubic

$$2\Delta\alpha'\beta'\gamma'-cp_3\alpha'\beta'\gamma-bp_2\alpha'\gamma'\beta-ap_1\beta'\gamma'\alpha=0, \quad (52)$$

where 2Δ is the area of ABC and p_1, p_2, p_3 are the perpendiculars from A, B, C respectively on $B'C'$, $C'A'$, $A'B'$. This cubic circumscribes the two triangles and furthermore passes through the intersection of corresponding sides, viz., the points AB, $A'B'$, etc. From the latter fact it can be readily deduced by means of the arguments, viz., the elliptic integrals which correspond to points on the curve, that A', B', C' must be corresponding points to A, B, C of the same system. Now, we know otherwise that the lines joining a point P of the curve to three pairs of corresponding points are in involution, so that in the same case the cubic can be written

$$AC'.BA'.CB'=AB'.BC'.CA', \quad (53)$$

where AC', etc., mean the areas of the triangles PAC', etc. Hence, when the curve is circular, we can find two relations connecting the two triangles. For, if we have the cubic $\alpha\beta\gamma=k\delta\epsilon\zeta$, then substituting the coordinates of the circular points, we get $k=1$, $a+\beta+\gamma=\delta+\epsilon+\zeta$, where a is the angle which α makes with a fixed line, etc., so that (53) gives $AC'.BA'.CB'=AB'.BC'.CA'$, where AC' is

the length of the line joining A, C', etc. Also, the sum of the angles between AC' and CA', BA' and AB', and CB' and BC' vanishes. Again, if the lines joining P to A, B, C respectively meet $B'C'$, $C'A'$, $A'B'$ in three points which form a triangle of given area M, the locus of P is the cubic

$$U = 2M(\alpha' - p_1)(\beta' - p_2)(\gamma' - p_3), \tag{54}$$

where U is the cubic (52). This represents a cubic passing through the points A, B, C.

Again, I consider a system of quadrilaterals with a given triangle of centres, and I seek the locus of the vertices when the circle passing through three of them cuts orthogonally a given circle. Taking the triangle of centres as the triangle of reference, and using areal coordinates, the equation of a circle is

$$(x + y + z)(lx + my + nz) = x^2 \cot A + y^2 \cot B + z^2 \cot C,$$

and expressing that this circle passes through the points $-x', y', z'$; $x', -y', z'$; $x', y', -z'$, its equation becomes

$$\begin{aligned}(x'^2 \cot A + y'^2 \cot B + z'^2 \cot C)(x + y + z)\{(y' + z' - x')x \\ + (z' + x' - y')y + (x' + y' - z')z\} \\ = (y' + z' - x')(z' + x' - y')(x' + y' - z')(x^2 \cot A + y^2 \cot B + z^2 \cot C).\end{aligned} \tag{55}$$

Hence, we see that if this circle cuts orthogonally a fixed circle, the locus of the fourth vertex (x', y', z') of the quadrilateral is a general cubic curve passing through the six imaginary points where the lines joining the middle points of the sides of the triangle of centres meet the polar circle. Also, the locus of one of the three vertices is a circular cubic passing through the four points where two of the lines joining the middle points of the sides meet the polar circle. We thus have as loci of the vertices of the quadrilateral one general cubic and three circular cubics.

I consider here now a locus connected with the cubic. Let AB be a chord of the cubic perpendicular to an asymptote, and let α, β, γ be the coordinates of a point P with regard to the triangle A, B, C, where C is the point at infinity on the same asymptote, then the equation of the cubic can be written

$$(l\alpha + m\beta)\gamma^2 + \alpha\beta(l'\alpha + m'\beta) + \gamma(a\alpha^2 + b\beta^2 + h\alpha\beta) = 0. \tag{56}$$

Now, if α', β', γ' are the coordinates of the intersection of the perpendiculars of the triangle PAB, it can easily be shown that $\alpha\beta' = \beta\alpha' = \gamma\gamma'$. Hence, the

locus of the intersection of the perpendiculars of the triangle ABP is the cubic

$$\alpha\beta\,(l\alpha + m\beta) + \gamma^2\,(l'\alpha + m'\beta) + \gamma\,(a\alpha^2 + b\beta^2 + h\alpha\beta) = 0, \tag{57}$$

intersecting the given cubic at the point at infinity C and at six points on the circle $\gamma^2 - \alpha\beta = 0$, namely, the circle described on AB as diameter. Let $l = l'$, $m = m'$, then the cubic has, it is easy to see, two mutually perpendicular asymptotes, and the intersection of perpendiculars of the triangle PAB lies on the curve.

Further, I note the following generation of a cubic. Given the base AB of a triangle, if a point P dividing in a given ratio the line joining the centre of the circumscribed circle and the intersection of the perpendiculars lies on a hyperbola with an asymptote perpendicular to AB, then the locus of the vertex C is a cubic such that AB is a chord of the cubic perpendicular to an asymptote at the finite point where that asymptote meets the curve.

Taking the axis as the base, and perpendicular to it at the middle point, the coordinates of the point P are given by

$$x = \frac{mx'}{m+n}, \quad y = \frac{m\,(x'^2 + y'^2 - c^2) + 2n\,(c^2 - x'^2)}{2y'\,(m+n)}, \tag{58}$$

where x', y' are the coordinates of the vertex and m/n is the given ratio. Hence, if P lies on a hyperbola with an asymptote perpendicular to AB, the locus of P is the cubic

$$\{my^2 + (m - 2n)(x^2 - c^2)\}(Ax + B) + y\,(Cx^2 + Dx + E), \tag{59}$$

which is such that A, B are two points on the curve on a perpendicular to the asymptote $Ax + B = 0$, where it meets the curve.

MIDDLETOWN, N. Y.

The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane.

Part Second: Complete Form-System of Invariants.*

BY HERBERT ELLSWORTH SLAUGHT.

§1.—INVARIANTS OF THE BINARY QUINTIC FORM.

1. Three invariants of the quintic as given by Salmon† are:

$$\left.\begin{aligned} P^2 &= \Pi\,(12)^2, \\ A &= \Sigma\,(12)^4(34)^2(35)^2(45)^2, \\ C &= P^2\Sigma\,(12)^{-4}(34)^{-2}(35)^{-2}(45)^{-2}, \end{aligned}\right\} \qquad (1)$$

where (ij) means the root difference

$$(\alpha_i - \alpha_j), \qquad (i, j = 1 \ldots . 5).$$

2. The ratio $A : P$ written in cross-ratio form is

$$\left.\begin{aligned} A : P = {} & [3214][3415][3512] + [4123][4325][4521] \\ & + [5132][5234][5431] + [5213][5314][5412] \\ & - [2314][2415][2513] - [3124][3425][3521] \\ & - [4132][4235][4531] - [5142][5243][5341] \\ & - [5123][5324][5421] - [4213][4315][4512] \end{aligned}\right\} \qquad (2)$$

in which

$$[ijkl] = \frac{(\alpha_i - \alpha_k)(\alpha_j - \alpha_l)}{(\alpha_j - \alpha_k)(\alpha_i - \alpha_l)}, \qquad (i, j, k, l = 1 \ldots . 5).$$

If we put in (2) $\qquad \lambda_1 = [4235],$

* The first part of this memoir is to be found in the *American Journal of Mathematics*, vol. XXII, pp. 343-388, 1900. It will be referred to here simply as Part First.

† "Modern Higher Algebra," third edition, §241.

and let $\lambda_2 \ldots \ldots \lambda_5$ be derived from λ_1 by successive application of the cyclic permutation (12345) to the indices, and make use of the relations*

$$\lambda_3 = \frac{1-\lambda_1}{1-\lambda_1\lambda_2}, \quad \lambda_4 = 1-\lambda_1\lambda_2, \quad \lambda_5 = \frac{1-\lambda_2}{1-\lambda_1\lambda_2}, \tag{3}$$

we get

$$\begin{aligned} A:P = -\{&(1-\lambda_1)^2[1+\lambda_2^2(1-\lambda_1)^2+\lambda_1^2\lambda_2^4] + (1-\lambda_2)^2[1+\lambda_1^2(1-\lambda_2)^2+\lambda_1^4\lambda_2^2] \\ &+(1-\lambda_1\lambda_2)^2[\lambda_1^2+\lambda_2^2+(1-\lambda_1\lambda_2)^2+(1-\lambda_1)^2(1-\lambda_2)^2]\} \\ &:\{\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)(1-\lambda_1\lambda_2)\}. \end{aligned} \tag{4}$$

3. We now identify the roots α_i with the variables† ν_i in the following order:

$$\left.\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \nu_5 & \nu_2 & \nu_3 & \nu_1 & \nu_4 \end{matrix}\right\} \tag{5}$$

and obtain the relation

$$\lambda_1 = \rho, \quad \lambda_2 = \frac{\sigma-1}{\sigma-\rho}. \tag{6}$$

Substituting (6) in (4) and changing to homogeneous coordinates

$$\rho = \frac{z_1}{z_3}, \quad \sigma = \frac{z_2}{z_3},$$

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*M. J. M. Hill, "The Anharmonic Ratios of the Roots of a Quintic" (Proceedings of the London Mathematical Society, vol. XIV, p. 182).

† E. H. Moore, "The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions" (*American Journal of Mathematics*, vol. XXII, p. 280, 1900). For the case $n=5$, I use the variables $\nu_1 \ldots \ldots \nu_5$, and the fundamental system of cross-ratios,

$$\rho \equiv \rho_4 = [\nu_1\nu_2\nu_3\nu_4] \text{ and } \sigma \equiv \rho_5 = [\nu_1\nu_2\nu_3\nu_5].$$

Whence, ρ_1, ρ_2, ρ_3 give the special values ∞, 0, 1.

With this notation, for example, the transformation T, corresponding to the substitution on the indices (15)(34), is derived as follows:

$$\rho' = \rho_{5243} = \frac{(\rho_5-\rho_4)(\rho_2-\rho_3)}{(\rho_2-\rho_4)(\rho_5-\rho_3)} = \frac{\rho-\sigma}{\rho(1-\sigma)},$$

$$\sigma' = \rho_{5241} = \frac{(\rho_5-\rho_4)(\rho_2-\rho_1)}{(\rho_2-\rho_4)(\rho_5-\rho_1)} = \frac{\rho-\sigma}{\rho},$$

which, in homogeneous coordinates, becomes

$$T; \quad z_1' : z_2' : z_3' = z_3(z_1-z_2) : (z_1-z_2)(z_3-z_2) : z_1(z_3-z_2).$$

See Part First, Arts. 4, 28 and 58.

there results

$$\left.\begin{aligned} A:P = -\{&\Sigma z_1^2(z_2-z_3)^4+\Sigma z_1^4(z_2-z_3)^2+\Sigma z_1^2z_2^2(z_1-z_2)^2 \\ &+(z_1-z_2)^2(z_1-z_3)^2(z_2-z_3)^2\}:\{z_1z_2z_3(z_1-z_2)(z_1-z_3)(z_2-z_3)\}. \end{aligned}\right\} \quad (7)$$

In a similar manner,

$$\left.\begin{aligned} C:P^3 = -\{&\Sigma z_1^4z_2^4(z_1-z_2)^2(z_1-z_3)^4(z_2-z_3^4)+\Sigma z_1^2z_2^4z_3^4(z_1-z_2)^4(z_1-z_3)^4 \\ &+\Sigma z_1^2z_2^2z_3^4(z_1-z_2)^2(z_1-z_3)^4(z_2-z_3)^4+z_1^4z_2^4z_3^4(z_1-z_2)^2(z_1-z_3)^2(z_2-z_3)^2\} \\ &:\{z_1^3z_2^3z_3^3(z_1-z_2)^3(z_1-z_3)^3(z_2-z_3)^3\}. \end{aligned}\right\} \quad (8)$$

4. Applying to (7) and (8) the transformation

$$z_1:z_2:z_3 = y_1-y_4:y_2-y_4:y_3-y_4, \quad (9)$$

we find the functions proportional to A, P^2, C;

$$\left.\begin{aligned} \delta_1A &= \Sigma(y_1-y_2)^2(y_3-y_4)^4+\Sigma(y_1-y_2)^2(y_1-y_3)^2(y_2-y_3)^2, \\ \delta_2P^2 &= (y_1-y_2)^2(y_1-y_3)^2(y_1-y_4)^2(y_2-y_3)^2(y_2-y_4)^2(y_3-y_4)^2, \\ \delta_3C &= \Sigma(y_1-y_2)^2(y_1-y_3)^4(y_1-y_4)^4(y_2-y_3)^4(y_2-y_4)^4 \\ &\qquad +P^2\,\Sigma(y_1-y_4)^2(y_2-y_4)^2(y_3-y_4)^2. \end{aligned}\right\} \quad (10)$$

5. It is desirable to evaluate A, P^2 and C in terms of the elementary symmetric functions:

$$-p_1=\Sigma y_i, \quad p_2=\Sigma y_iy_j, \quad -p_3=\Sigma y_iy_jy_k, \quad p_4=y_1y_2y_3y_4, \quad (11)$$
$$(i,j,k=1\ \ldots\ 4).$$

The results for A and P^2 are easily found:

$$\left.\begin{aligned} \delta_1A &= 3^2p_3^2+2\cdot 3p_2^3+2^3\cdot 5p_2p_4+\ \ldots\ \text{terms containing } p_1 \text{ as a factor}, \\ \delta_2P^2 &= 2^8p_4^3-3^3p_3^4-2^2p_2^3p_3^2-2^7p_2^2p_4^2+2^4p_2^4p_4+2^4\cdot 3^2p_2p_3^2p_4 \\ &\qquad +\ \ldots\ \text{terms containing } p_1 \text{ as a factor}. \end{aligned}\right\} \quad (12)$$

6. For the first part of C, whose weight is 18, we have

$$\left.\begin{aligned} \Sigma(y_1-y_2)^2(y_1-y_3)^4(y_1-y_4)^4(y_2-y_4)^4(y_3-y_4)^4 \equiv\ & a_1p_2^9+a_2p_3^6 \\ &+a_3p_3^2p_4^3+a_4p_2^6p_3^2+a_5p_2^3p_3^4+a_6p_2^7p_4+a_7p_2^5p_4^2+a_8p_2^3p_4^3 \\ &+a_9p_2p_4^4+a_{10}p_2p_3^4p_4+a_{11}p_2^4p_3^2p_4+a_{12}p_2^2p_3^2p_4^2 \\ &+41 \text{ terms containing } p_1 \text{ as a factor}. \end{aligned}\right\} \quad (13)$$

7. In (13) put

$$y_1=-y_3=1, \qquad y_2=y_4=0,$$

Whence by (11), $p_1 = p_3 = p_4 = 0, \quad p_2 = -1.$

From which $a_1 = -2^2.$

To determine a_2, a_4, a_5, put in succession in (13),

$$y_1, y_2, y_3, y_4 = 0, 2, -1, -1;\; 0, -3, 2, 1;\; 0, 4, -3, -1.$$

whence

$$p_1, p_2, p_3, p_4 = 0, -3, -2, 0;\; 0, -7, 6, 0;\; 0, -13, -12, 0.$$

These substitutions lead to the conditions,

1) $2^4 a_2 + 3^6 a_4 - 2^2 \cdot 3^3 a_5 = -2 \cdot 3^8,$
2) $2^4 \cdot 3^6 a_2 + 3^2 \cdot 7^6 a_4 - 2^2 \cdot 3^4 \cdot 7^3 a_5 = 2^{11} \cdot 5^5 + 3^4 \cdot 5^5 + 2^{11} \cdot 3^4 \cdot 13 - 7^9,$
3) $2^{10} \cdot 3^6 a_2 + 2^2 \cdot 3^2 \cdot \overline{13}^6 a_4 - 2^6 \cdot 3^4 \cdot \overline{13}^3\, a_5 = 3^4 5^5 \cdot 7^4 + 2^{11} \cdot 7^4 \cdot 17 + 2^{11} \cdot 3^4 \cdot 5^6 - \overline{13}^9,$

from which

$$a_2 = -2 \cdot 3^5, \quad a_4 = -2 \cdot 3 \cdot 11, \quad a_5 = -2^2 \cdot 3^2 \cdot 11.$$

8. To find a_6, a_7, a_8, a_9, put in succession,

$$y_1, y_2, y_3, y_4 = 1, -1, 1, -1;\; 1, -1, 2, -2;\; 1, -1, 3, -3;\; 2, -2, 3, -3$$

Whence,

$$p_1, p_2, p_3, p_4 = 0, -2, 0, 1;\; 0, -5, 0, 4;\; 0, -10, 0, 9;\; 0, -13, 0, 36.$$

Then from the equations:

1) $2^6 a_6 + 2^4 a_7 + 2^2 a_8 + a_9 = 2^{10},$
2) $5^6 a_6 + 2^2 \cdot 5^4 a_7 + 2^4 \cdot 5^2 a_8 + 2^6 \cdot a_9 = -2^6 \cdot 11 \cdot 29 \cdot 113,$
3) $2^6 \cdot 5^6 a_6 + 2^4 \cdot 3^2 \cdot 5^4 a_7 + 2^2\; 3^4 \cdot 5^2 a_8 + 3^6 a_9 = -2^{10} \cdot 83,687,$
4) $\overline{13}^6 a_6 + 2^2 \cdot 3^2 \cdot \overline{13}^4 \cdot a_7 + 2^4 \cdot 3^4 \cdot \overline{13}^2 a_8 + 2^6 \cdot 3^6 \cdot a_9 = -2^6 \cdot 7,238,879,$

we get,

$$a_6 = 2^6, \quad a_7 = -2^7 \cdot 11, \quad a_8 = -2^{10} \cdot 7, \quad a_9 = 2^{10} \cdot 47.$$

9. To find a_3, a_{10}, a_{11}, a_{12}, put in succession,

$$y_1, y_2, y_3, y_4 = 1, 1, 1, -3;\; 1, 1, 2, -4;\; 2, 2, -3, -1;\; 1, 1, 4, -6,$$

which give,

$$p_1, p_2, p_3, p_4 = 0, -6, 8, -3;\; 0, -11, 18, -8;\; 0, -9, 4, 12;\; 0, -27, 50, -24.$$

Then from the equations:

$$\begin{aligned}
&1)\quad 3a_3 - 2^7 a_{10} + 2^4\cdot 3^3 a_{11} - 2^2\cdot 3^2 a_{12} = 2^7\cdot 3^3\cdot 7\cdot 11,\\
&2)\quad 2^6 a_3 - 2^2\cdot 3^4\cdot 11 a_{10} + \overline{11}^4 a_{11} - 2^3\cdot \overline{11}^2 a_{12} = 2^3\cdot 1{,}339{,}307,\\
&3)\quad 2^4 a_3 - 2^4 a_{10} + 3^6 a_{11} + 2^2\cdot 3^2 a_{12} = 2^3\cdot 3^2\cdot 11\cdot 617,\\
&4)\quad 2^6 a_3 - 2^2\cdot 3\cdot 5^4 a_{10} + 3^{10} a_{11} - 2^3\cdot 3^5 a_{12} = 2^3\cdot 3^2\cdot 7\cdot 11\cdot 9{,}767,
\end{aligned}$$

we find,

$$a_3 = -2^7\cdot 3^2\cdot 23, \quad a_{10} = 2^4\cdot 3^3, \quad a_{11} = 2^3\cdot 131, \quad a_{12} = 2^5\cdot 3^2\cdot 5.$$

Thus we have found:

$$\left.\begin{aligned}
&\Sigma (y_1-y_2)^2(y_1-y_3)^4(y_1-y_4)^4(y_2-y_3)^4)(y_2-y_4)^4 \equiv -2^2 p_2^9 - 2\cdot 3^5 p_3^6\\
&\quad -2^7\cdot 3^2\cdot 23 p_3^2 p_4^3 - 2\cdot 3\cdot 11 p_2^6 p_3^2 - 2^2\cdot 3^2\cdot 11 p_2^3 p_3^4 + 2^6 p_2^7 p_4 - 2^7\cdot 11 p_2^5 p_4^2\\
&\quad -2^{10}\cdot 7 p_2^3 p_4^3 + 2^{10}\cdot 47 p_2 p_4^4 + 2^4\cdot 3^3 p_2 p_3^4 p_4 + 2^3\cdot 131 p_2^4 p_3^2 p_4 + 2^5\cdot 3^2\cdot 5 p_2^2 p_3^2 p_4^2\\
&\quad + \ldots\ldots \text{ terms containing } p_1 \text{ as a factor.}
\end{aligned}\right\} \quad (14)$$

10. If the second part of C be evaluated in a similar manner and the result combined with (14), we have finally,

$$\left.\begin{aligned}
&\delta_3 C = -2^2 p_2^9 - 2\ 3^3\cdot 23 p_3^6 - 2.17 p_2^6 p_3^2 - 2^7\cdot 151 p_3^2 p_4^3\\
&\quad -2^2\cdot 73 p_2^3 p_3^4 - 2^6 p_2^7 p_4 + 2^7 p_2^5 p_4^2 - 2^{10}\cdot 13 p_2^3 p_4^3\\
&\quad + 2^{10}\cdot 5\cdot 11 p_2 p_4^4 + 2^4\cdot 3^2\cdot 5^2 p_2 p_3^4 p_4 + 2^3\cdot 3^3 p_2^4 p_3^2 p_4 + 2^5\cdot 7\cdot 11 p_2^2 p_3^2 p_4^2\\
&\quad + \ldots\ldots \text{ terms containing } p_1 \text{ as a factor.}
\end{aligned}\right\} \quad (15)$$

§2.—Invariants of the Subgroup $G_{24}^{(1)}$.

Critical and Non-Critical Points. Arts. 11–13.

11. One of the points

$$1i \qquad (i = 2 \ldots\ldots 5)$$

is *non-critical* in the following cases:

(a). In general, for all the linear transformations of G_{120} which form the subgroup $G_{24}^{(1)}$ and which *permute* these four points among themselves in 4! ways.

(b). In particular, for a certain dihedron subgroup of $G_{24}^{(1)}$, which leaves the corresponding *point fixed.* Thus the point $1i$ is fixed under

$$G_6^{1i} \sim \{jkl\} \text{ all}, \qquad (i, j, k, l = 2 \ldots\ldots 5).$$

(c). For such quadratic transformations as leave the *point fixed*, thus $1i$ is fixed under each of the 6 quadratic transformations of the set [Part First, Art. 10],

$$D_{1i}^{-1} \sim \{jkl\} \text{ all } (1i).$$

These belong to the set S_{1i}^{-1}, where

$$S_{1i} \sim \{2345\} \text{ all } (1i) \qquad \text{[Part First, Art. 15]}$$

is the complete set of transformations through any one of which $G_{24}^{(1)}$ is transformed into $G_{24}^{(i)}$.

12. Under *all other quadratic transformations* the point $1i$ is *critical;* that is, it must be regarded as a pencil of directions which goes into a range of points on one of the fundamental sides. [Part First, Art. 12.] These transformations consist of

(a) the remaining 18 in the set S_{1i}^{-1}.

(b) all of the sets, S_{1j}^{-1}, $(j \neq i = 2 \ldots . 5)$.

In all, $3 \cdot 24 + 18 = 90$.

13. *A pencil,* $1i$, *is invariant* under such transformations as permute among themselves its infinity of direction tangents. Evidently this will happen if, and only if, the *corresponding point is fixed.*

As just shown, the point $1i$ is fixed under the *linear* transformations G_6^{1i} and under the *quadratic* transformations of the set D_{1i}^{-1}, and hence the pencil $1i$ is invariant under the linear subgroup,

$$G_6^{1i} \sim \{jkl\} \text{ all},$$

and under the *quadratic subgroup,*

$$G_{12}^{1i} \sim \{jkl\} \text{ all } \{1i\}.*$$

This is in agreement with Part First, Art. 33, where the 4 pencils and 6 sides were found to form a system of 10 conjugate elements under G_{120}.

The linear subgroup G_6^{1i}, which plays an important rôle in the sequel, is also a subgroup of the quadratic group $G_{24}^{(i)}$, its transformations being the *only linear* transformations in that subgroup.

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* See foot-note to Part First, Art. 10.

The Complete Form-System for $G_{24}^{(1)}$. Arts. 14–17.

14. The linear subgroup $G_{24}^{(1)}$ is projectively connected with Klein's collineation group $G_{4!}$ by the transformation*

$$\left.\begin{array}{ll} \text{direct;} & z_1:z_2:z_3 = x_2+x_3:x_3+x_1:x_1+x_2, \\ \text{inverse;} & x_1:x_2:x_3 = -z_1+z_2+z_3:z_1-z_2+z_3:z_1+z_2-z_3, \end{array}\right\} \quad (1)$$

where $x_1 \ldots x_3$ are homogeneous point coordinates; or by the transformation†

$$\left.\begin{array}{lll} \text{direct;} & z_1:z_2:z_3 & = y_1-y_4:y_2-y_4:y_3-y_4, \\ \text{inverse;} & y_1:y_2:y_3:y_4 & = 3z_1-z_2-z_3:-z_1+3z_2-z_3 \\ & & \quad :-z_1-z_2+3z_3:-z_1-z_2-z_3, \end{array}\right\} \quad (2)$$

where $y_1 \ldots y_4$ are supernumerary homogeneous point coordinates.

15. The complete form-system of invariants of Klein's collineation group G_{4} consists of the elementary symmetric functions

$$\Sigma y_i y_j, \quad \Sigma y_i y_j y_k, \quad \Pi y_i, \qquad (i, j, k = 1 \ldots 4) \quad (3)$$

with the identical relation

$$\Sigma y_i = 0.$$

Hence, the complete form-system of invariants of $G_{24}^{(1)}$ will be derived from that of $G_{4!}$ by applying the transformation (2) to the forms (3). The results are:

$$\left.\begin{array}{rll} p_2 = \Sigma y_i y_j & = & -6\Sigma z_1^2 + 4\Sigma z_1 z_2 \\ -p_3 = \Sigma y_i y_j y_k & = & 8\Sigma z_1^3 - 8\Sigma z_1^2 z_2 + 16 z_1 z_2 z_3, \\ p_4 = \Pi y_i & = & -3\Sigma z_1^4 + 4 z_1^3 z_2 - 20\Sigma z_1^2 z_2 z_3 + 14\Sigma z_1^2 z_2^2, \end{array}\right\} \quad (4)$$

where again $\Sigma y_i = 0$ identically in the z's.

---

*Part First, (5), Art. 7.

†For the case $G_{4!}^{\prime}$, I introduce supernumerary coordinates conveniently in the form

$$y_1:y_2:y_3:y_4 = -x_1+x_2+x_3:x_1-x_2+x_3:x_1+x_2-x_3:-x\;\;-x_2-x_3,$$

from which, in combination with (1), we derive (2). It thus appears that

$$\Sigma y_i = 0$$

identically in the z's as well as in the x's. I thus have the y's related to the z's of $G_{24}^{(1)}$ just as Professor Moore has related the y's to the x's of $G_{4!}$. See his paper, "Concerning Klein's Group of $(n+1)!$ n-ary Collineations" (*American Journal of Mathematics*, vol. XXII, pp. 336–342, 1900).

In these forms put

$$\Sigma z_1 = p, \quad \Sigma z_1 z_2 = q, \quad z_1 z_2 z_3 = r, \tag{5}$$

and they become

$$\left.\begin{aligned} p_2 &= -2\,[3p^2 - 2^3 q] \\ -p_3 &= \quad 2^3\,[\,p^3 - 2^2 pq + 2^3 r], \\ p_4 &= -p\,[3p^3 - 2^4 pq + 2^6 r]. \end{aligned}\right\} \tag{6}$$

16. The forms A, P^2, C, initially given as functions of the roots of the quintic, have been interpreted (a) as functions of the homogeneous coordinates of the transformations of the cross-ratio group G_{120}, and hence of the subgroup $G_{24}^{(1)}$, by the agreement (5), Art. 3; (b) as functions of the supernumerary homogeneous coordinates of the transformations of $G_{4!}$ by virtue of the substitutions (9), Art. 4, since this is the same as (2), Art. 14; and (c) as rational integral functions of p_1, p_2, p_3, p_4 by (12), Art. 5, and (15), Art. 10.

Hence, by Art. 15, they are ***absolute invariants*** of $G_{24}^{(1)}$. They may be further simplified by substituting in them the value of p_2, p_3, p_4 from (6), Art. 15, and remembering that $p_1 = 0$, thus

$$\delta_1 A = 2p^2q^2 - 2\cdot 3\,(p^3 r + q^3) + 19pqr - 3^2 r^2, \tag{7}$$

$$\delta_2 P^2 = p^2q^2r^2 - 2^2\,(p^3r^3 + q^3r^2) + 2\cdot 3^2 pqr^3 - 3^3 r^4, \tag{8}$$

$$\left.\begin{aligned} \delta_3 C = {} & 2^2\cdot 5^2 p^4q^4r^2 - 2\cdot 3^3\cdot 23 r^6 + 2^4\cdot 5\cdot 7\, p^3q^3r^3 - 2\cdot 5^2\cdot 43 p^2q^2r^4 \\ & + 2\cdot 3^2\cdot 157 pqr^5 - 2\cdot 143\,(p^2q^5r^2 + p^5q^2r^3) - 2\cdot 17\,(p^6r^4 + q^6r^2) \\ & - 2^2\cdot 73\,(p^3r^5 + q^3r^4) + (p^2q^8 + p^8q^2r^2) + 2\cdot 5\cdot 53(p^4qr^4 + pq^4r^3) \\ & + 2\cdot 5^2\,(p^7qr^3 + pq^7r) - 2^2\,(p^9r^3 + q^9) - 2^2\cdot 3\,(p^6q^3r^2 + p^3q^6r). \end{aligned}\right\} \tag{9}$$

17. The form P^2 is of special interest later.

In terms of $y_1 \ldots\ldots y_4$ for the group $G_{4!}$, we have by (10), Art. 4,

$$P_y = \Pi\,(y_1 - y_2) \equiv \sqrt{\Delta}, \tag{10}$$

and in terms of $z_1 \ldots\ldots z_3$ for the group $G_{24}^{(1)}$, by (7), Art. 3,

$$P_z = z_1 z_2 z_3\,(z_1 - z_2)(z_1 - z_3)(z_2 - z_3), \tag{11}$$

in which latter the factors on the right give the six sides of the quadrangle [Fig. II, Part First].

By well-known principles,* the linear homogeneous substitution group $G_{n!}$, which is isomorphic with the symmetric permutation group on n letters, has one, and only one, *fundamental relative invariant*, namely, $\sqrt{\Delta}$, since (except for $n = 4$) it has one, and only one, self-conjugate subgroup $G_{\frac{n!}{2}}$.

Since the index of this subgroup is 2, it follows that the only relative invariants under $G_{n!}$ $(n \neq 4)$ must throw off the factor (-1).

This conclusion, however, holds also for $n = 4$, since $G_{4!}$ can be generated by transformations *all of period* 2,

K', L', M', [Part First, (3), Art. 6.]

so that if any primitive root of unity higher than the second could be thrown off, it would have to be built out of the factors $(+1)$ and (-1), which is impossible.

Hence, P_y is the only fundamental relative invariant under $G_{4!}$, from which it follows that P_z is the only such form under $G_{24}^{(1)}$.

Therefore, all relative invariant forms under $G_{24}^{(1)}$ must be of the form

$$P^{2\kappa+1} \cdot f(p_2, p_3, p_4), \tag{12}$$

where f is a rational integral function.

§3.—Characteristics of Invariants Under G_{120}.

Fundamental Notions and Definitions. Arts. 18–21.

18. Since a quadratic transformation, when applied to any function of the z's, must double its degree, it follows that no such function can be *invariant in the ordinary sense under* G_{120}.

The only invariant form possible is a *rational fraction* such that, under any quadratic transformation of the group, a common factor (a function of the z's) is thrown off in numerator and denominator.

Evidently, the degree of the numerator must be the same as that of the denominator and equal to that of the factor thrown off.

19. Since a linear transformation does not change the degree of the function operated upon, it follows that numerator and denominator of an invariant fraction must each be an absolute or relative invariant under $G_{24}^{(1)}$.

* Weber, "Lehrbuch der Algebra," vol. II, p. 161–164.

Such a fraction cannot have both terms relative invariants under $G_{24}^{(1)}$, for [Art. 17, (12)] it would have the form

$$\frac{p^{2\kappa_1+1}\cdot\phi_1(p_2,\, p_3,\, p_4)}{p^{2\kappa_2+1}\cdot\phi_2(p_2,\, p_3,\, p_4)},$$

which reduces, according as $\kappa_1 \gtrless \kappa_2$, to

$$\text{(I)}\quad \frac{p^{2(\kappa_1-\kappa_2)}\cdot\phi_1(p_2,\, p_3,\, p_4)}{\phi_2(p_2,\, p_3,\, p_4)} \quad\text{or (II)}\quad \frac{\phi_1(p_2,\, p_3,\, p_4)}{p^{2(\kappa_2-\kappa_1)}\cdot\phi_2(p_2,\, p_3,\, p_4)},$$

in which both numerator and denominator are absolute invariants under $G_{24}^{(1)}$. [Art. 16.]

If, then, either numerator or denominator alone is a relative invariant under $G_{24}^{(1)}$, the fraction will have one of the forms

$$\text{(III)}\quad \frac{p^{2\kappa_1+1}\cdot\phi_1(p_2,\, p_3,\, p_4)}{\phi_2(p_2,\, p_3,\, p_4)},\quad \text{(IV)}\quad \frac{\phi_1(p_2,\, p_3,\, p_4)}{p^{2\kappa_2+1}\cdot\phi_2(p_2,\, p_3,\, p_4)}.$$

Forms of the types (III) and (IV) will be considered at the end of the paper [Art. 52].

In the succeeding investigation, the forms considered for numerator and denominator of invariant fractions will be absolute invariants under $G_{24}^{(1)}$, so that every such fraction will be of the type

$$\text{(V)}\quad \frac{\phi_1(p_2,\, p_3,\, p_4)}{\phi_2(p_2,\, p_3,\, p_4)}.$$

20. It follows at once that an invariant fraction cannot have its numerator and denominator of odd degree in z_1, z_2, z_3, for it would then have the form

$$\frac{p_3^{2\kappa_1+1}\cdot\phi_1(p_2,\, p_3^2,\, p_4)}{p_3^{2\kappa_2+1}\cdot\phi_2(p_2,\, p_3^2,\, p_4)},$$

since p_3 is the only fundamental invariant form of *odd* degree in z_1, z_2, z_3 under the linear subgroup $G_{24}^{(1)}$.

But such a fraction, when reduced, becomes, according as $\kappa_1 \gtrless \kappa_2$,

$$\text{(VI)}\quad \frac{p_3^{2(\kappa_1-\kappa_2)}\cdot\phi_1(p_2,\, p_3^2,\, p_4)}{\phi_2(p_2,\, p_3^2,\, p_4)} \quad\text{or (VII)}\quad \frac{\phi_1(p_2,\, p_3^2,\, p_4)}{p_3^{2(\kappa_2-\kappa_1)}\cdot\phi_2(p_2,\, p_3^2,\, p_4)},$$

in which both terms are of *even* degree in z_1, z_2, z_3.

Thus, *numerator and denominator of an invariant fraction must be rational, integral functions of* p_2, p_3^2, p_4, *and hence also of* p, q, r, *and, therefore, symmetric functions of* z_1, z_2, z_3 owing to the relations (5) and (6), Art. 15.

21. From these considerations come the following definitions:

(a). *An invariant under a quadratic operator of* G_{120} *is a fraction whose numerator and denominator are rational integral functions of* z_1, z_2, z_3 *such that it is transformed into itself by the operator, after throwing off a factor in the z's common to numerator and denominator.*

(b). *An invariant under the Group* G_{120} *is a fraction which is invariant under every one of a system of generators of the group, and so under every transformation of the group.* For this purpose we use the generators of $G_{24}^{(1)}$,

$$K \sim (34), \quad L \sim (23)(45), \quad M \sim (45),$$

and as the extender to G_{120}, the quadratic inversion,

$$T' \sim (12), \qquad \text{[Arts. 4, 28, Part First.]}$$

(c). Any homogeneous function of z_1, z_2, z_3, which is suitable to form the numerator or denominator of an invariant fraction as above defined, is called an *invariant form.* Evidently such a form may be composed of factors or terms,* each of which is a simpler *invariant form.*

(d). Those forms by means of which all other invariant forms of the group can be rationally and integrally expressed, are called the *system of fundamental forms,* or the *complete form-system* of the group.

Theorems on Invariant Forms. Arts. 22–25.

Theorem I.

22. *The most general invariant form under* G_{120} *is of degree* $6n$ *in* z_1, z_2, z_3 *and throws off the factor* r^{2n} *under the quadratic generator* T'.

Proof. We denote by $f_s(z_1, z_2, z_3)$ any homogeneous function of degree s in z_1, z_2, z_3, which is invariant under G_{120} and investigate the value of s and the nature of the factor thrown off when f_s is operated upon by the quadratic extender,

$$T' \sim (12)\,; \; z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2. \qquad (1)$$

*In the case of invariant terms, of course all must have the same "throw-off."

Thus we have* $$f_s(z_1, z_2, z_3)_{T'} = f_{2s}(z_2z_3, z_1z_3, z_1z_2). \tag{2}$$

Since f_s is an invariant form by hypothesis, some factor of degree s [Art. 18] in z_1, z_2, z_3, must be thrown off, thus

$$f_{2s}(z_2z_3, z_1z_3, z_1z_2) = f_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3). \tag{3}$$

Now apply T' again to (3) and as the left becomes homogeneous of degree $4s$, we have

$$(z_1z_2z_3)^s \cdot f_s(z_1, z_2, z_3) = f_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3)_{T'}. \tag{4}$$

Hence, dividing by $f_s(z_1, z_2, z_3)$,

$$(z_1z_2z_3)^s, = r^s, = \phi_s(z_1, z_2, z_3)\,\phi_s(z_1, z_2, z_3)_{T'}. \tag{5}$$

Therefore, $$\phi_s(z_1, z_2, z_3) = r^t \qquad (t \leqq s) \tag{6}$$

and $$\phi_s(z_1, z_2, z_3)_{T'} = r^{2t}. \tag{7}$$

Substituting (6) and (7) in (5),

$$r^s = r_{3t}, \tag{8}$$

giving $$s = 3t. \tag{9}$$

Then $$s \equiv 0 \pmod 3.$$

But $$s \equiv 0 \pmod 2. \qquad \text{[Art. 20.]}$$

Hence, $$s \equiv 0 \pmod 6. \tag{10}$$

Then from (6),

$$r^t = \phi_s = \phi_{6n} = (z_1z_2z_3)^t,$$

from which it follows, $$t = 2n. \tag{11}$$

Therefore, the general invariant form, R, is of degree $6n$, and throws off the factor r^{2n} under T'.

Theorem II.

23. *The most general invariant form may be decomposed into two factors,*

$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)},$$

where μ equals zero or a positive integer and $R_{6(n-2\mu)}$ contains no factor of P.

---

*For convenience, the operator is here written as a subscript to the operand.

Proof: The six factors of P are the six sides of the quadrangle Π_1. [(11), Art. 17.] Since these form a conjugate system under $G_{24}^{(1)}$, and since R_{6n} is invariant in the ordinary sense under all transformations of $G_{24}^{(1)}$, it follows that if R_{6n} contains *any* factor of P, it must contain *every* such factor, and if any of these factors are repeated all must be repeated equally often.

Moreover, since we are considering only such forms as are *absolute* invariants under $G_{24}^{(1)}$, such a factor cannot involve P to an *odd* power. [Art. 19.]

Hence, R_{6n} contains as a factor either no factor of P or else $P^{2\mu}$, where μ is a positive integer.

Theorem III.

24. *The curve*
$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)} = 0, \tag{1}$$
has a multiple point of order $2(n+\mu)$ *at each of the critical points* $1i$, $(i = 2 \ldots . 5)$.

Proof: Consider the two factors of (1).

(a). $P^{2\mu}$ fulfills all the conditions for an *invariant form*, namely:

It is an absolute invariant under the linear subgroup $G_{24}^{(1)}$. [Art. 16.]

It is of requisite degree in z_1, z_2, z_3; that is, $6 \cdot 2\mu$ according to theorem I.

It throws off the proper factor under T', namely, $r^{2 \cdot 2\mu}$, thus
$$P_{T'}^{2\mu} = [z_1^2 z_2^2 z_3^2 (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2]^\mu = r^{4\mu} \cdot P^{2\mu}.$$
The curve
$$P = 0$$
is a degenerate sextic, having three branches through each of the points $1i$, since it represents the six sides of the quadrangle Π_1. [Art. 17.] Hence,
$$P^{2\mu} = 0$$
has at each of these points a 6μ-ple point.

(b). Hence, the remaining factor
$$R_{6(n-2\mu)}$$
must be an *invariant form*, since the product R_{6n} is such by hypothesis. That is,
$$[R_{6(n-2\mu)}]_{T'} = r^{2(n-2\mu)} \cdot R_{6(n-2\mu)}. \tag{2}$$
From this it follows that the curve
$$R_{6(n-2\mu)} = 0 \tag{3}$$
has the multiple points $1i$ each of order $2(n - 2\mu)$.

For if the curve (3) *contains any of the points* $1i$, *it must contain all of them*, since they form a conjugate system under $G_{24}^{(1)}$, under which subgroup, of course, $R_{6(n-2\mu)}$ is invariant.

25. *That it does contain three of these points* follows from the properties of the quadratic transformation T', whose critical points are the three coordinate vertices and whose critical lines form the triangle of reference [Art. 10, Part First], namely:

Under T' a curve through one vertex goes into another curve through the same vertex and throws off as an extra factor the opposite coordinate side; and, conversely, a curve under T' can throw off a coordinate side as a factor only when it contains the opposite coordinate vertex [special case of Art. 11, Part First].

Now (3) reproduces itself under T' and throws off the factor

$$r^{2(n-2\mu)}$$

which contains all three coordinate sides. Hence the curve contains *each of the coordinate vertices and, therefore, all four critical points* as just shown.

Since a curve passing *once* through each of the coordinate vertices throws off, under T', precisely the factor r, in order to throw off $r^{2(n-2\mu)}$, a curve must have $2(n-2\mu)$ branches through each vertex.

Thus *three* (and hence all *four*) of the critical points are multiple points of order $2(n-2\mu)$ on the curve (3).

Therefore, the curve (1)

$$P^{2\mu}\cdot R_{6(n-2\mu)}=0$$

has each of these points as a multiple point of order

$$[6\mu+2(n-2\mu)]=2(n+\mu). \tag{4}$$

§4.—COMPLETE DETERMINATION OF R_{6n} FOR $n=1, 2, 3$.

General Forms of Degree 6, 12, 18. Arts. 26–35.

26. The determination of the most general invariant forms of any given degree $6n$ suitable for numerator and denominator of invariant fractions, involves three steps.

(1). Set up the most general form of the given degree invariant under $G_{24}^{(1)}$, involving arbitrary coefficients.

(2). Apply the transformation T' by which $G_{24}^{(1)}$ extends to G_{120}.

(3). Determine the arbitrary coefficients in such a way that the required factor, r^{2n}, may divide out and leave the original form.

27. For this purpose, all forms will be expressed in terms of p, q, r, since the application of T' to these is peculiarly simple, namely,

$$\left.\begin{aligned} p_{T'} &= q, \\ q_{T'} &= pr, \\ r_{T'} &= r^2. \end{aligned}\right\} \qquad (1)$$

28. The most general invariant form of degree 6 under $G_{24}^{(1)}$ is

$$a_1 p_2^3 + a_2 p_3^2 + a_3 p_2 p_4,$$

which may be expressed in terms of p, q, r in the following manner:*

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	p^6	p^4q	q^3	p^3r	pqr	r^2	p^2q^2
-2^3a_1	3^3	$-2^3\cdot 3^3$	-2^9	0	0	0	$2^6\cdot 3^2$
$+2^6a_2$	1	-2^3	0	2^4	-2^6	2^6	2^4
$+2a_3$	3^2	$-2^3\cdot 3^2$	0	$2^6\cdot 3$	-2^9	0	2^7
Apply T'	q^6	pq^4r	p^3r^3	q^3r^2	pqr^3	r^4	$p^2q^2r^2$
Divide by r^2	..	....	p^3r	q^3	pqr	r^2	p^2q^2

*In each case the form is written in a rectangular array with the original letters at the top, those resulting from the application of T' next to the bottom, those left after dividing all terms possible by r^{2n} in the bottom row, while the arbitrary coefficients occupy the column at the left and the numerical coefficients are written in the body of the array.

The conditions to be met are—

(1). The new form must throw off r^2.

(2). The resulting quotient must be the same as the original form.

In order to meet these conditions,

(a). The coefficients of p^6 and p^4q must vanish.

(b). The coefficients of like terms in the original and resulting forms may be equated [r^2 having been divided out].

These give only *two independent relations*,

$$2^2\cdot 3^3a_1 - 2^5a_2 - 3^2a_3 = 0,$$
$$2^5a_1 - 2^3a_2 - 3a_3 = 0.$$

From which $$a_1 : a_2 : a_3 = 6 : 9 : 40.$$

Hence the most general invariant of the 6th degree is proportional to

$$2\cdot 3p_2^3 + 3^2p_3^2 + 2^3\cdot 5p_2p_4. \tag{2}$$

This is precisely the form A derived from the quintic. [(12), Art. 5.]

Its value in terms of p, q, r may be read from the table

$$\delta A = 2p^2q^2 - 2\cdot 3\,(p^3r + q^3) + 19pqr - 3^2r^2, \tag{3}$$

thus agreeing with (7), Art. 16.

29. The most general invariant form of degree 12 under $G_{24}^{(1)}$ is

$$b_1p_2^6 + b_2p_3^4 + b_3p_4^3 + b_4p_2^4p_4 + b_5p_3^2 + b_6p_2^2p_4^2 + b_7p_2p_3^2p_4, \tag{4}$$

which, in terms of p, q, r in rectangular array, is

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(9)	(9)
	p^{12}	p^8q^2	$p^{10}q$	p^6q^3	p^9r	p^7qr	q^3r^2	p^3r^3	p^2q^5
$+2^6b_1$	3^6	$2^6\cdot3^5\cdot5$	$-2^4\cdot3^6$	$-2^{11}\cdot3^3\cdot5$	0	0	0	0	$-2^{16}\cdot3^2$
$+2^{12}b_2$	1	$2^5\cdot3$	-2^4	-2^8	2^5	$-2^7\cdot3$	0	2^{11}	0
$-b_3$	3^3	$2^8\cdot3^2$	$-2^4\cdot3^3$	-2^{12}	$2^6\cdot3^3$	$-2^{11}\cdot3^2$	0	0	0
-2^4b_4	3^5	$2^7\cdot3^3\cdot7$	$-2^4\cdot3^5$	$-2^{13}\cdot3^2$	$2^6\cdot3^4$	$-2^{11}\cdot3^3$	0	0	-2^{16}
-2^9b_5	3^3	$2^4\cdot3^2\cdot19$	$-2^4\cdot3^3$	$-2^7\cdot67\cdot$	$2^4\cdot3^3$	$-2^6\cdot3^4$	-2^{15}	0	-2^{13}
$+2^2b_6$	3^4	$2^6\cdot3^2\cdot13$	$-2^4\cdot3^4$	$-2^{11}\cdot3^2$	$2^7\cdot3^3$	$-2^{12}\cdot3^2$	0	0	0
$+2^7b_7$	3^2	$2^4\cdot53$	$-2^4\cdot3^2$	$-2^7\cdot17$	$2^4\cdot3\cdot7$	$-2^6\cdot59$	0	$2^{12}\cdot3$	0
Apply T'	q^{12}	$p^2q^8r^2$	$pq^{10}r$	$p^3q^6r^3$	q^2r^2	pq^7r^3	p^3r^7	q^3r^6	$p^5q^2r^5$
Divide by r^4	....	....	....	....	....	....	p^3r^3	q^3r^2	p^5q^2r

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$p^5q\,r$	p^4qr^2	pq^4r	q^6	p^6r^2	pqr^3	p^4q^4	$p^2q^2r^2$	p^3q^3r	r^4
0	0	0	2^{18}	0	0	$2^{12}\cdot3^3\cdot5$	0	0	0
$2^9\cdot3$	$-2^{10}\cdot3$	0	0	$2^7\cdot3$	-2^{13}	2^8	$2^{11}\cdot3$	-2^{11}	2^{12}
$2^{14}\cdot3$	$-2^{16}\cdot3$	0	0	$2^{12}\cdot3^2$	0	0	0	0	0
$2^{13}\cdot3^3$	0	2^{18}	0	0	0	$2^{12}\cdot3^3$	0	$-2^{17}\cdot3$	0
$2^9\cdot3^2\cdot5$	$-2^9\cdot3^3$	2^{15}	0	$2^6\cdot3^3$	0	$2^{10}\cdot13$	$2^{12}\cdot3^2$	$-2\cdot\cdot11$	0
$2^{13}\cdot3\cdot5$	$-2^{16}\cdot3$	0	0	$2^{12}\cdot3^2$	0	2^{14}	2^{18}	-2^{17}	0
$2^9\cdot3^3$	$-2^9\cdot7^2$	0	0	$2^6\cdot3\cdot19$	-2^{15}	2^{11}	$2^{13}\cdot5$	-2^{14}	0
$p^2q^5r^4$	pq^4r^5	p^4qr^6	p^6r^6	q^6r^4	pqr^7	$p^4q^4r^4$	$p^2q^2r^6$	$p^3q^3r^5$	r^8
p^2q^5	pq^4r	p^4qr^2	p^6r^2	q^6	pqr^3	p^4q^4	$p^2q^2r^2$	p^3q^3r	r^4

30. The conditions to be met here are—

(a). The coefficients of terms (1) to (6) must vanish.

(b). The coefficients of term (7), (8); (9), (10); (11), (12); (13), (14), must be equal in pairs.

(c). The coefficients of terms (15 to (19) are the same in the old and new forms, and hence give no relations.

There are then 10 relations, which are not all independent, but readily reduce to the following 7 equations:

b_1	b_2	b_3	b_4	b_5	b_6	b_7		
$2^6 \cdot 3^6$	$+ 2^{12}$	$- 3^3$	$- 2^4 \cdot 3^5$	$- 2^9 \cdot 3^3$	$+ 2^2 \cdot 3^4$	$+ 2^7 \cdot 3^2$	$= 0$	(5)
$2^4 \cdot 3^5$	$+ 2^9 \cdot 3$	$- 3^2$	$- 2^3 \cdot 3^3 \cdot 7$	$- 2^5 \cdot 3^2 \cdot 19$	$+ 3^2 \cdot 13$	$+ 2^3 \cdot 53$	$= 0$	
$2^5 \cdot 3^5 \cdot 5$	$+ 2^8$	$- 1$	$- 2^5 \cdot 3^2$	$- 2^4 \cdot 67$	$+ 2 \cdot 3^2$	$+ 2^2 \cdot 17$	$= 0$	
2^{12}	$- 2^7 \cdot 3$	$+ 3^2$	0	$+ 2^3 \cdot 3^3$	$- 2^2 \cdot 3^2$	$- 2 \cdot 3 \cdot 19$	$= 0$	
0	$+ 2^5$	$- 1$	0	$- 2^6$	0	$+ 2 \cdot 3$	$= 0$	
$2^8 \cdot 3^2$	$+ 2^7 \cdot 3$	$- 3$	$- 2^3 \cdot 5 \cdot 7$	$- 2^4 \cdot 61$	$+ 2 \cdot 3 \cdot 5$	$+ 2^2 \cdot 3^3$	$= 0$	
0	$- 2^6 \cdot 3$	$+ 3$	$+ 2^6$	$+ 2^2 \cdot 7 \cdot 13$	$- 2^2 \cdot 3$	$- 7^2$	$= 0$	

31. Two invariants of the 12th degree are already known, P^2 [Art. 24], and A^2 [Art. 28].

Hence their values,

	p_2^6	p_3^4	p_4^3	$p_2^4p_4$	$p_2^2p_3^2$	$p_2^2p_2^4$	$p_2p_3^2p_4$
$A^2 =$	$2^2 \cdot 3^2$	$+ 3^4$	0	$+ 2^5 \cdot 3 \cdot 5$	$+ 2^3 \cdot 3^3$	$+ 2^6 \cdot 5^2$	$+ 2^4 \cdot 3^2 \cdot 5$
$P^2 =$	0	$- 3^3$	$+ 2^8$	$+ 2^4$	$- 2^2$	$- 2^7$	$+ 2^4 \cdot 3^2$
	b_1	b_2	b_3	b_4	b_5	b_6	b_7

(6)

must each satisfy the above system of equations. This is easily verified. Therefore, there exists an infinity of solutions of the system (5) of the form

$$m_1 A^2 + m_2 P^2, \tag{7}$$

where m_1 and m_2 are arbitrary parameters. Hence, all the first minors in the determinant of (5) must vanish, as well as the determinant itself. If, now, any second minor does not vanish, then no solution exists other than those included in the form (7). The second minor obtained by omitting the 5th and 7th columns and the 1st and 2d rows is easily shown to be different from zero.

Hence, all invariant forms of the 12th degree are included in the form (7).

32. The most general form of degree 18 invariant under $G_{24}^{(1)}$ involves 12 arbiträry constants, thus

$$\begin{aligned} c_1 p_2^9 + c_2 p_3^6 + c_3 p_2^6 p_3^2 + c_4 p_3^2 p_4^3 + c_5 p_2^3 p_3^4 + c_6 p_2^7 p_4 + c_7 p_2^5 p_4^2 + c_8 p_2^3 p_4^3 \\ + c_9 p_2 p_4^4 + c_{10} p_2 p_3^4 p_4 + c_{11} p_2^4 p_3^2 p_4 + c_{12} p_2^2 p_3^2 p_4^2. \end{aligned} \tag{8}$$

When expressed in terms of p, q, r in the rectangular array, this becomes

	(1)	(2)	(3)	(4)	(5)	(6)
	p^{18}	$p^{16}q$	$p^{15}r$	$p^{14}q^2$	$p^{12}q^3$	$p^{12}r^2$
-2^9c_1	3^9	$-2^3\cdot 3^{10}$	0	$2^8\cdot 3^9$	$-2^{11}\cdot 3^7\cdot 7$	0
$+2^{18}c_2$	1	$-2^3\cdot 3$	$2^4\cdot 3$	$2^4\cdot 3\cdot 5$	$-2^8\cdot 5$	$2^6\cdot 3\cdot 5$
$+2^{12}c_3$	3^6	$-2^3\cdot 3^7$	$2^4\cdot 3^6$	$2^4\cdot 3^5\cdot 47$	$-2^9 . 3^5\cdot 5$	$2^6\cdot 3^6$
-2^6c_4	3^3	$-2^3\cdot 3^4$	$2^4\cdot 3^3\cdot 5$	$2^4\cdot 3^2\cdot 43$	$-2^8\cdot 5\cdot 23$	$2^6\cdot 3^2\cdot 5\cdot 23$
$-2^{15}c_5$	3^3	$-2^3\cdot 3^4$	$2^5\cdot 3^3$	$2^5\cdot 3^2\cdot 23$	$-2^9\cdot 73$	$2^7\cdot 3^4$
$+2^7c_6$	3^8	$-2^3\cdot 3^9$	$2^6\cdot 3^7$	$2^6\cdot 3^6\cdot 5\cdot 7$	$-2^9\cdot 3^5\cdot 7\cdot 11$	0
-2^5c_7	3^7	$-2^3\cdot 3^8$	$2^7\cdot 3^6$	$2^7\cdot 3^5\cdot 17$	$-2^{10}\cdot 3^4\cdot 5\cdot 7$	$2^{12}\cdot 3^5$
$+2^3c_8$	3^6	$-2^3\cdot 3^7$	$2^6\cdot 3^6$	$2^6\cdot 3^5\cdot 11$	$-2^9\cdot 3^5\cdot 7$	$2^{12}\cdot 3^5$
$-2c_9$	3^5	$-2^3\cdot 3^6$	$2^8\cdot 3^4$	$2^{11}\cdot 3^3$	$-2^{12}\cdot 3^2\cdot 7$	$2^{13}\cdot 3^4$
$-2^{13}c_{10}$	3^2	$-2^3\cdot 3^3$	$2^5\cdot 3\cdot 5$	$2^5\cdot 67$	$-2^{10}\cdot 11$	$2^7\cdot 3\cdot 5^2$
$-2^{10}c_{11}$	3^5	$-2^3\cdot 3^6$	$2^4\cdot 3^4\cdot 7$	$2^4\cdot 3^3\cdot 137$	$-2^8\cdot 3^2\cdot 143$	$2^6\cdot 3^{14}\cdot 19$
$+2^8c_{12}$	3^4	$-2^3\cdot 3^5$	$2^4\cdot 3^3\cdot 11$	$2^4\cdot 3^2\cdot 7\cdot 19$	$-2^8\cdot 3^2\cdot 43$	$2^6\cdot 3^2\cdot \overline{13}^2$
Apply T'	q^{18}	$pq^{16}r$	$q^{15}r^2$	$p^2q^{14}r^2$	$p^3q^{12}r^3$	$q^{12}r^4$
Divide by r^6	....	......	......	....	....	....

	(7) $p^{10}q^4$	(8) p^8q^5	(9) $p^{13}qr$	(10) $p^{11}q^2r$	(11) $p^{10}qr^2$
-2^9c_1	$2^{13}\cdot 3^7\cdot 7$	$-2^{16}\cdot 3^6\cdot 7$	0	0	0
$+2^{18}c_2$	$2^8\cdot 3\cdot 5$	$-2^{11}\cdot 3$	$-2^6\cdot 3\cdot 5$	$2^9\cdot 3\cdot 5$	$-2^{10}\cdot 3^5$
$+2^{12}c_3$	$2^{10}\cdot 3^3\cdot 5\cdot 29$	$-2^{20}\cdot 3^2$	$-2^6\cdot 3^6\cdot 5$	$2^{13}\cdot 3^5$	$-2^{10}\cdot 3^6$
-2^6c_4	$2^{12}\cdot 17$	-2^{16}	$-2^6\cdot 3^2\cdot 71$	$2^{11}\cdot 3\cdot 47$	$-2^{10}\cdot 3\cdot 7\cdot 43$
$-2^{15}c_5$	$2^8\cdot 491$	$-2^{11}\cdot 3\cdot 41$	$-2^7\cdot 3^3\cdot 5$	$2^9\cdot 3^2\cdot 31$	$-2^{11}\cdot 3^4$
$+2^7c_6$	$2^{12}\cdot 3^5\cdot 5\cdot 7$	$-2^{15}\cdot 3^3\cdot 7\cdot 13$	$-2^9\cdot 3^6\cdot 7$	$2^{12}\cdot 3^6\cdot 7$	0
-2^5c_7	$2^{12}\cdot 3^3\cdot 5\cdot 17$	$-2^{15}\cdot 3^2\cdot 61$	$-2^{10}\cdot 3^5\cdot 7$	$2^{15}\cdot 3^4\cdot 5$	$-2^{15}\cdot 3^4\cdot 5$
$+2^3c_8$	$2^{13}\cdot 3^3\cdot 11$	$-2^{17}\cdot 3^3$	$-2^9\cdot 3^5\cdot 7$	$2^{12}\cdot 3^4\cdot 19$	$-2^{15}\cdot 3^4\cdot 5$
$-2c_9$	$2^{16}\cdot 3^2$	-2^{19}	$-2^{11}\cdot 3^3\cdot 7$	$2^{15}\cdot 3^4$	$-2^{16}\cdot 3^3\cdot 5$
$-2^{13}c_{10}$	$2^8\cdot 3\cdot 43$	$-2^{11}\cdot 5^2$	$-2^7\cdot 73$	$2^9\cdot 3\cdot 47$	$-2^{11}\cdot 71$
$-2^{10}c_{11}$	$2^{11}\cdot 3^2\cdot 59$	$-2^{15}\cdot 5\cdot 13$	$-2^6\cdot 3^3\cdot 7\cdot 11$	$2^{11}\cdot 3^3\cdot 5^2$	$-2^{10}\cdot 3^3\cdot 53$
$+2^8c_{12}$	$2^{10}\cdot 277$	$-2^{15}\cdot 13$	$-2^6\cdot 3^2\cdot 157$	$2^{12}\cdot 3\cdot 5\cdot 11$	$-2^{10}\cdot 3\cdot 11\cdot 41$
Apply T'	$p^4q^{10}r^4$	$p^5q^8r^5$	$pq^{13}r^3$	$p^2q^{11}r^4$	$pq^{10}r^5$
Divide by r^6	....	....	....	....	....

	(12)	(13)	(14)	(15)	(16)	(17)	(18)
	p^9q^3r	q^9	p^9r^3	p^6r^4	q^6r^2	p^4q^7	p^7q^4r
-2^9c_1	0	-2^{27}	0	0	0	$-2^{23}\cdot 3^4$	0
$+2^{18}c_2$	$-2^{11}\cdot 3\cdot 5$	0	$2^{11}\cdot 5$	$2^{12}\cdot 3\cdot 5$	0	0	$2^{12}\cdot 3\cdot 5$
$+2^{12}c_3$	$-2^{12}\cdot 3\cdot 5\cdot 17$	0	0	0	2^{24}	$-2^{20}\cdot 11\cdot$	$2^{16}\cdot 3^4\cdot 5$
-2^6c_4	$-2^{14}\cdot 5\cdot 11$	0	$2^{12}\cdot 5\cdot 47$	$2^{18}\cdot 5^2$	0	0	2^{20}
$-2^{15}c_5$	$-2^{11}\cdot 5\cdot 61$	0	$2^{11}\cdot 3^3$	$2^{12}\cdot 3^3$	0	-2^{17}	$2^{14}\cdot 3\cdot 31$
$+2^7c_6$	$-2^{15}\cdot 3^4\cdot 5\cdot 7$	0	0	0	0	$-2^{21}\cdot 3^2\cdot 5$	$2^{18}\cdot 3\cdot 5\cdot 7$
-2^5c_7	$-2^{17}\cdot 3^4\cdot 5$	0	0	0	0	-2^{23}	$2^{19}\cdot 3^2\cdot 5$
$+2^3c_8$	$-2^{15}\cdot 3^3\cdot 5^2$	0	$2^{18}\cdot 3^3$	0	0	0	$2^{22}\cdot 3^2$
$-2c_9$	$-2^{19}\cdot 3\cdot 5$	0	$2^{20}\cdot 3^2$	$2^{24}\cdot 3$	0	0	2^{23}
$-2^{13}c_{10}$	$-2^{11}\cdot 3^3\cdot 5$	0	$2^{11}\cdot 3^2\cdot 5$	$2^{12}\cdot 3\cdot 5\cdot 7$	0	0	2^{19}
$-2^{10}c_{11}$	$-2^{13}\cdot 3\cdot 5\cdot 47$	0	$2^{12}\cdot 3^4$	0	0	-2^{20}	$2^{16}\cdot 5\cdot 41$
$+2^8c_{12}$	$-2^{12}\cdot 5\cdot \overline{11}^2$	0	$2^{13}\cdot 3^2\cdot 11$	$2^{18}\cdot 3^2$	0	0	$2^{18}\cdot 17$
Apply T'	$p^3q^9r^5$	p^9r^9	q^9r^6	q^6r^8	p^6r^{10}	$p^7q^4r^7$	$p^4q^7r^6$
Divide by r^6	....	p^9r^3	q^9	q^6r^2	p^6r^4	p^7q^4r	p^4q^7

	(19)	(20)	(21)	(22)	(23)	(24)
	p^3r^5	q^3r^4	p^2q^8	$p^8q^2r^2$	p^7qr^3	pq^7r
-2^9c_1	0	0	$2^{24}\cdot 3^3$	0	0	0
$+2^{18}c_2$	$2^{16}\cdot 3$	0	0	$2^{11}\cdot 3^2\cdot 5$	$-2^{12}\cdot 3\cdot 5$	0
$+2^{12}c_3$	0	0	2^{22}	$2^{12}\cdot 3^5\cdot 5$	0	-2^{24}
-2^6c_4	2^{24}	0	0	$2^{14}\cdot 3^2\cdot 29$	$-2^{17}\cdot 67$	0
$-2^{15}c_5$	0	-2^{21}	0	$2^{11}\cdot 3^3\cdot 19$	$-2^{13}\cdot 3^4$	0
$+2^7c_6$	0	0	2^{26}	0	0	-2^{27}
-2^5c_7	0	0	0	$2^{19}\cdot 3^3\cdot 5$	0	0
$+2^3c_8$	0	0	0	$2^{18}\cdot 3^5$	$-2^{21}\cdot 3^3$	0
$-2c_9$	0	0	0	$2^{22}\cdot 3^2$	$-2^{23}\cdot 3^2$	0
$-2^{13}c_{10}$	$2^{18}\cdot 3$	0	0	$2^{11}\cdot 3\cdot 7\cdot 19$	$-2^{13}\cdot 3\cdot 41$	0
$-2^{10}c_{11}$	0	0	0	$2^{13}\cdot 3^4\cdot 13$	$-2^{17}\cdot 3^3$	0
$+2^8c_{12}$	0	0	0	$2^{12}\cdot 5\cdot 353$	$-2^{19}\cdot 3\cdot 5$	0
Apply T'	q^3r^{10}	p^3r^7	$p^8q^2r^8$	$p^2q^8r^6$	pq^7r^7	p^7qr^9
Divide by r^6	q^3r^4	p^3r	$p^8q^2r^2$	p^2q^8	pq^7r	p^7qr^3

	(25)	(26)	(27)	(28)	(29)	(30)	(31)
	$p^6q^3r^2$	p^3q^6r	$p^5q^2r^3$	$p^2q^5r^2$	p^4qr^4	pq^4r^3	r^6
-2^9c_1	0	0	0	0	0	0	0
$+2^{18}c_2$	$-2^{14}\cdot 3\cdot 5$	0	$2^{15}\cdot 3\cdot 5$	0	$-2^{15}\cdot 3\cdot 5$	0	2^{18}
$+2^{12}c_3$	$-2^{17}\cdot 3^3\cdot 5$	$2^{23}\cdot 5$	0	$-2^{22}\cdot 3^2$	0	0	0
-2^6c_4	$-2^{18}\cdot 5^2$	0	$2^{20}\cdot 19$	0	$-2^{22}\cdot 7$	0	0
$-2^{15}c_5$	$-2^{14}\cdot 3\cdot 67$	2^{20}	$2^{16}\cdot 3^2\cdot 5$	$-2^{20}\cdot 3$	$-2^{15}\cdot 3^3$	2^{22}	0
$+2^7c^6$	0	$2^{24}\cdot 3\cdot 7$	0	0	0	0	0
-2^5c_7	$-2^{22}\cdot 3^2\cdot 5$	2^{26}	0	-2^{27}	0	0	0
$+2^3c_8$	$-2^{21}\cdot 3^2\cdot 7$	0	$2^{24}\cdot 3^2$	0	0	0	0
$-2c_9$	$-2^{24}\cdot 3$	0	2^{27}	0	-2^{27}	0	0
$-2^{13}c_{10}$	$-2^{14}\cdot 3\cdot 41$	0	$2^{16}\cdot 5\cdot 11$	0	$-2^{15}\cdot 89$	0	0
$-2^{10}c_{11}$	$-2^{23}\cdot 3$	2^{23}	$2^{11}\cdot 3^3$	$-2^{22}\cdot 5$	0	2^{24}	0
$+2^{18}c_{12}$	$-2^{17}\cdot 5^3$	0	$2^{19}\cdot 47$	0	$-2^{22}\cdot 3$	0	0
Apply T'	$p^3q^6r^7$	$p^6q^3r^8$	$p^2q^5r^8$	$p^5q^2r^9$	pq^4r^9	p^4qr^{10}	r^{12}
Divide by r^6	p^3q^6r	$p^6q^3r^2$	$p^2q^5r^2$	$p^5q^2r^3$	pq^4r^3	p^4qr^4	r^6

	(32)	(33)	(34)	(35)	(36)	(37)
	p^6q^6	p^5q^5r	$p^4q^4r^2$	$p^3q^3r^3$	$p^2q^2r^4$	pqr^5
-2^9c_1	$2^{20}\cdot 3^4\cdot 7$	0	0	0	0	0
$+2^{18}c_2$	2^{12}	$-2^{14}\cdot 3$	$2^{14}\cdot 3\cdot 5$	$-2^{17}\cdot 5$	$2^{16}\cdot 3\cdot 5$	$-2^{18}\cdot 3$
$+2^{12}c_3$	$2^{16}\cdot 211$	$-2^{18}\cdot 3^2\cdot 19$	$2^{18}\cdot 3\cdot 5$	0	0	0
-2^6c_4	0	0	0	0	0	0
$-2^{15}c_5$	$2^{14}\cdot 17$	$-2^{17}\cdot 3\cdot 5$	$2^{17}\cdot 3\cdot 13$	$-2^{19}\cdot 11$	$2^{18}\cdot 3^2$	0
$+2^7c_6$	$2^{18}\cdot 3^2\cdot 7^2$	$-2^{21}\cdot 3^3\cdot 7$	0	0	0	0
-2^5c_7	$2^{21}\cdot 3^2$	$-2^{22}\cdot 3\cdot 11$	$2^{24}.3.5$	0	0	0
$+2^3c_8$	2^{21}	$-2^{23}\cdot 3$	$2^{25}\cdot 3$	-2^{27}	0	0
$-2c_9$	0	0	0	0	0	0
$-2^{13}c_{10}$	2^{15}	$-2^{17}\cdot 3$	$2^{18}\cdot 7$	-2^{22}	$2^{19}\cdot 3^2$	-2^{21}
$-2^{10}c_{11}$	$2^{16}\cdot 5\cdot 7$	$-2^{18}\cdot 3^2\cdot 7$	$2^{18}\cdot 139$	$-2^{23}\cdot 3$	0	0
$+2^8c_{12}$	2^{18}	$-2^{20}\cdot 3$	$2^{20}\cdot 13$	$-2^{23}\cdot 3$	2^{24}	0
Apply T'	$p^6q^6r^6$	$p^5q^5r^7$	$p^4q^4r^8$	p^3q^3r	$p^2q^2r^{10}$	pqr^{11}
Divide by r^6	p^6q^6	p^5q^5r	$p^4q^4r^2$	$p^3q^3r^3$	$p^2q^2r^4$	pqr^5

33. The conditions in this case are—

(a). The coefficients of terms (1) to (12) must vanish.

(b). The coefficients of terms (13) to (30) must be equal in pairs; that is,

(13), (14); (15), (16); .... (29), (30).

(c). The coefficients of terms (31) to (37) are the same in the new form as in the original, and hence afford no relations.

These conditions give rise to 21 relations among the 12 c's. This system reduces readily to the following 16 equations:

	c_1	c_2	c_3	c_4	c_5	c_6
(1)	$2^8 \cdot 3^9$	-2^{17}	$-2^{11} \cdot 3^6$	$+2^5 \cdot 3^3$	$+2^{14} \cdot 3^3$	$-2^6 \cdot 3^8$
(2)	0	$+2^{13}$	$+2^7 \cdot 3^5$	$-2 \cdot 3^2 \cdot 5$	$-2^{11} \cdot 3^2$	$+2^4 \cdot 3^6$
(3)	$2^8 \cdot 3^7 \cdot 7$	$-2^{14} \cdot 5$	$-2^8 \cdot 3^3 \cdot 157$	$+2^2 \cdot 5 \cdot 23$	$+2^{12} \cdot 73$	$-2^4 \cdot 3^5 \cdot 7 \cdot 11$
(4)	0	$+2^{12} \cdot 5$	$+2^6 \cdot 3^5$	$-3 \cdot 5 \cdot 23$	$-2^{10} \cdot 3^3$	0
(5)	$2^7 \cdot 3^6 \cdot 7$	$-2^{11} \cdot 3$	$-2^{14} \cdot 3^2$	$+2^4$	$+2^8 \cdot 3 \cdot 41$	$-2^4 \cdot 3^3 \cdot 7 \cdot 13$
(6)	$-2^6 \cdot 3^7 \cdot 7$	$+2^{10} \cdot 3 \cdot 5$	$+2^6 \cdot 3^3 \cdot 5 \cdot 29$	$-2^2 \cdot 17 \cdot$	$-2^7 \cdot 491$	$+2^3 \cdot 3^5 \cdot 5 \cdot 7$
(7)	0	$-2^{12} \cdot 3 \cdot 5$	$-2^6 \cdot 3^6$	$+3 \cdot 7 \cdot 43$	$+2^{10} \cdot 3^4$	0
(8)	-2^{18}	$+2^{11} \cdot 5$	0	$-5 \cdot 47$	$-2^8 \cdot 3^3$	0
(9)	0	$+2^6 \cdot 3 \cdot 5$	-2^{12}	-5^2	$-2^3 \cdot 3^3$	0
(10)	$-2^8 \cdot 3^4$	$+2^6 \cdot 3 \cdot 5$	$+2^4 \cdot 7 \cdot 83$	-2^2	$-2^5 \cdot 101$	$+2 \cdot 3^2 \cdot 5 \cdot 29$
(11)	0	$+2^4 \cdot 3$	0	-1	-2^6	0
(12)	$2^{13} \cdot 3^3$	$+2^9 \cdot 3^2 \cdot 5$	$+2^4 \cdot 191$	$-3^2 \cdot 29$	$-2^6 \cdot 3^3 \cdot 19$	-2^{12}
(13)	0	$-2^8 \cdot 3 \cdot 5$	$+2^{13}$	$+67$	$+2^5 \cdot 3^4$	$+2^{11}$
(14)	0	$-2^8 \cdot 3 \cdot 5$	$-2^5 \cdot 5 \cdot 7 \cdot 13$	$+5^2$	$+2^5 \cdot 5 \cdot 53$	$-2^7 \cdot 3 \cdot 7$
(15)	0	$-2^7 \cdot 3 \cdot 5$	$+2^8 \cdot 3^2$	-19	$-2^5 \cdot 3 \cdot 31$	0
(16)	0	$-2^5 \cdot 3 \cdot 5$	0	$+7$	$+2^2 \cdot 5 \cdot 31$	0

	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	
(1)	$+2^4\cdot 3^7$	$-2^2\cdot 3^6$	$+3^5$	$-2^{12}\cdot 3^2$	$+2^9\cdot 3^5$	$-2^7\cdot 3^4$	$=0$
(2)	$-2^3\cdot 3^8$	$+3^5$	-3^3	$+2^9\cdot 5$	$-2^5\cdot 3^3\cdot 7$	$+2^3\cdot 3^2\cdot 11$	$=0$
(3)	$+2^3\cdot 3^4\cdot 5\cdot 7$	$-3^5\cdot 7$	$+2\cdot 3^2\cdot 7$	$-2^{11}\cdot 11$	$+2^6\cdot 3^2\cdot 143$	$-2^4\cdot 3^2\cdot 43$	$=0$
(4)	$-2^5\cdot 3^4$	$+2^3\cdot 3^4$	$-2^2\cdot 3^3$	$+2^8\cdot 5^2$	$-2^4\cdot 3^3\cdot 19$	$+2^2 . 3\cdot \overline{13}^2$	$=0$
5)	$+2^2\cdot 3^2\cdot 61$	$-2^2\cdot 3^3$	$+2^2$	$-2^6\cdot 5^2$	$+2^7\cdot 5\cdot 13$	$-2^5\cdot 13$	$=0$
(6)	$-2\cdot 3^3\cdot 5\cdot 17$	$+3^3\cdot 11$	$-2\cdot 3^2$	$+2^5\cdot 3\cdot 43$	$-2^5\cdot 3^2\cdot 59$	$2^2\cdot 277$	$=0$
(7)	$+2^4\cdot 3^4 . 5$	$-2^2\cdot 3^4\cdot 5$	$+2\cdot 3^3\cdot 5$	$-2^8\cdot 71$	$+2^4\cdot 3^3\cdot 53$	$-2^2\cdot 3\cdot 11\cdot 41$	$=0$
(8)	0	$+2^3\cdot 3^3$	$-2^3\cdot 3^2$	$+2^6\cdot 3^2\cdot 5$	$-2^4\cdot 3^4$	$+2^3\cdot 3^2\cdot 11$	$=0$
(9)	0	0	$-2\cdot 3$	$+2\cdot 3\cdot 5\cdot 7$	0	$+2^2\cdot 3^2$	$=0$
(10)	-241	$+2\cdot 3^2$	-1	$+2^8$	$-2^2\cdot 13\cdot 17$	$+2^2\cdot 17$	$=0$
(11)	0	0	0	$+2\cdot 3$	0	0	$=0$
(12)	$-2^4\cdot 3^3\cdot 5$	$+2^2\cdot 3^5$	$-2^3\cdot 3^2$	$+2^4\cdot 3\cdot 7\cdot 19$	$-2^3\cdot 3^4\cdot 13$	$+5\cdot 353$	$=0$
(13)	0	$-2\cdot 3^3$	$+2\cdot 3^2$	$-2^3\cdot 3\cdot 41$	$+2^4\cdot 3^3$	$-2^4\cdot 3\cdot 5$	$=0$
(14)	$+2^3\cdot 61$	$-3^2\cdot 7$	$+2\cdot 3$	$-2^3\cdot 3\cdot 41$	$+2^{11}$	$-2\cdot 5^3$	$=0$
(15)	-2^6	$+2\cdot 3^2$	-2^2	$+2^3\cdot 5\cdot 11$	$-2^3\cdot 67$	$+2\cdot 47$	$=0$
(16)	0	0	$+1$	-89	$+2^6$	$-2^2\cdot 3$	$=0$

34. Two invariant forms of degree 18 are already known, A^3 and AP^2, and since these depend upon two of the forms derived in (7), (8), (9), Art. 16, it is at once suggested that the third form C, which is of degree 18, may also belong to this system. [See (15), Art. 12.]

In fact, it is easily verified that the 16 equations are satisfied by these three forms:

	c_1	c_2	c_3	c_4	c_5	c_6
$A^3=$	$+\,2^3\cdot 3^3$	$+\,3^6$	$+\,2^2\cdot 3^5$	0	$+\,2\cdot 3^6$	$+\,2^5\cdot 3^3\cdot 5$
$AP^2=$	0	$-\,3^5$	$-\,2^3\cdot 3$	$+\,2^8\cdot 3^2$	$-\,2\cdot 3^2\cdot 11$	$+\,2^5\cdot 3$
$C=$	$-\,2^2$	$-\,2\cdot 3^3\cdot 23$	$-\,2\cdot 17$	$-\,2^7\cdot 151$	$-\,2^2\cdot 73$	$-\,2^6$
	p_2^9	p_3^6	$p_2^6 p_3^2$	$p_3^2 p_4^3$	$p_2^3 p_3^4$	$p_2^7 p_4$

c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
$+\,2^7 . 3^2\cdot 5^2$	$+\,2^9\cdot 5^3$	0	$+\,2^3\cdot 3^5\cdot 5$	$+\,2^5\cdot 3^4\cdot 5$	$+\,2^6\cdot 3^3\cdot 5^2$
$-\,2^7$	$-\,2^9 . 7$	$+\,2^{11}\cdot 5$	$+\,2^3\cdot 3^3$	$+\,2^4\cdot 53\cdot$	$+\,2^9\cdot 3^2$
$+\,2^7$	$-\,2^{10}\cdot 13$	$+\,2^{10}\cdot 5\cdot 11$	$+\,2^4\cdot 3^2\cdot 5^2$	$+\,2^3\cdot 3^3$	$+\,2^5\cdot 7\cdot 11$
$p_2^5 p_4^2$	$p_2^3 p_4^3$	$p_2 p_4^4$	$p_2 p_3^4 p_4$	$p_2^4 p_3^2 p_4$	$p_2^2 p_3^2 p_4^2$

(9)

35. It follows that an infinity of solutions of the above system of equations exists of the form

$$m_1 A^3 + m_2 AP^2 + m_3 C, \tag{10}$$

where the m's are arbitrary parameters.

Hence, all determinants of the 12th order in the matrix of the coefficients must vanish and also *all* the first and second minors of these. A third minor, however, is easily found which does not vanish, namely, by depleting the first 7 rows and the 2d, 5th and 10th columns. Thus, no solution exists other than those included in (10).

Therefore, *the only invariant forms of degree* 18 *are included in the general form*

$$m_1A^3 + m_2AP^2 + m_3C.$$

§5.—The System of Fundamental Invariants.

Invariants of the Linear Groups G_6^{1i}. Arts. 36–38.

36. It has been shown that each of the pencils $1i$ $(i = 2 \ldots . 5)$ is invariant under a dihedron group G_6^{1i}. [Art. 13.]

The known* complete form-systems of these subgroups will now furnish the means of determining the system of fundamental invariants for G_{120}.

These auxiliary systems may be read by the group properties already shown in the configuration Π, Fig. X, Part First.

It will be sufficient to deduce the system for one pencil, say at the vertex 13.

The three conics $\qquad 13 \cdot 24, \quad 13 \cdot 25, \quad 13 \cdot 45$

are permuted under the dihedron

$$G_6^{13} \sim \{245\} \text{ all.}$$

Hence, this group must also permute among themselves those directions in the pencil 13 which are given by the corresponding tangents to these conics at that point,

$$\left.\begin{aligned} z_1 - 2z_2 &= 0, \\ 2z_1 - z_2 &= 0, \\ z_1 + z_2 &= 0. \end{aligned}\right\} \tag{1}$$

The product of these tangential quantics is, therefore, one of the cubic invariant forms under G_6^{13},

$$f_1 = -2z_1^3 + 3(z_1^2z_2 + z_1z_2^2) - 2z_2^3. \tag{2}$$

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* Klein, "Ikosaeder," Kapitel I, §9.

The other cubic invariant is the product of the three sides of the quadrangle Π_1 which pass through the point 13,

$$f_2 = z_1^2 z_2 - z_1 z_2^2. \qquad (3)$$

37. In order to find the quadratic invariant under G_6^{13}, we take the invariant conic belonging to the fundamental system of $G_{24}^{(1)}$ [Art. 14],

$$p_2 \equiv -3\Sigma z_1^2 + 2\Sigma z_1 z_2 = 0 \qquad (4)$$

and find the tangents to it from the point 13, namely [Art. 46, Part First],

$$\left.\begin{aligned} \omega z_1 + z_2 &= 0, \\ \omega^2 z_1 + z_2 &= 0. \end{aligned}\right\} \qquad (5)$$

The product of these tangential quantics is, then, the quadratic invariant form of G_6^{13},

$$f_3 = z_1^2 - z_1 z_2 + z_2^2. \qquad (6)$$

This follows since, under G_6^{13}, *the point* 13 *is fixed and the conic is fixed*, so that the tangents must be either fixed or permuted, making their product an invariant. Since there is *only one quadratic invariant* under G_6^{13}, it must be f_3 thus determined.

In like manner the systems may be found for the other pencils and their groups.

38. It will be seen that the above forms f_1, f_2, f_3 correspond to Klein's forms F_1, F_2, F_3 in the following manner:

$$\left.\begin{array}{ccc} F_1 & F_2^2 & F_3^3 \\ f_1 & -3^3 f_2^2 & 2^2 f_3^3 \end{array}\right\} \qquad (7)$$

so that the identity* holds

$$F_1^2 - F_2^2 - F_3^3 \equiv f_1^2 + 3^3 f_2^2 - 2^2 f_3^3 = 0. \qquad (8)$$

For the point 13 we have

$$\left.\begin{aligned} f_1^2 &= 4(z_1^6 + z_2^6) - 12(z_1^5 z_2 + z_1 z_2^5) - 3(z_1^4 z_2^2 + z_1^2 z_2^4) + 26 z_1^3 z_2^3, \\ f_2^2 &= + (z_1^4 z_2^2 + z_2^2 z_1^4) - 2 z_1^3 z_2^3, \\ f_3^3 &= (z_1^6 + z_2^6) - 3(z_1^5 z_2 + z_1 z_2^5) + 6(z_1^4 z_2^2 + z_1^2 z_2^4) - 7 z_1^3 z_2^3. \end{aligned}\right\} \qquad (9)$$

From (9) we see at once that (8) holds.

* Klein, "Ikosaeder," p. 49.

The only remaining type-form of the 6*th degree under* G_6^{13} *is*

$$f_1 f_2 = -2(z_1^5 z_2 - z_1 z_2^5) + 5(z_1^4 z_2^2 - z_1^2 z_2^4),^* \qquad (10)$$

which alone is non-symmetric in z_1, z_2.

Relation of A, P^2, C *to* f_1, f_2, f_3. Arts. 39–47.

39. It has been seen by theorems I, II [Arts. 23, 24] that every invariant form under G_{120} may be reduced to

$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)},$$

and that it has at each of the critical points a multiple point of order $2(n+\mu)$.

For the forms already considered the parameters, n and μ have the following special values:

	n	μ	$2(n+\mu)$
A	1	0	2
A^2	2	0	4
A^3	3	0	6
P^2	2	1	6
AP^2	3	2	10
C	3	0	6

(1)

40. Consider the sextic curve

$$A = 0. \qquad (2)$$

* See Art. 46.

The product of its tangential quantics at 13, one of its double points, is the binary quadratic invariant for that point. For

$$A = 0$$

is fixed, and the point 13 is fixed under the linear group G_6^{13}. Therefore, the two tangents are either fixed or permuted, and their product is the quadratic invariant f_3, since only one such form exists under G_6^{13}.

In this sense the ternary form A is said to correspond to the binary form f_3, and since A is the only ternary invariant form of the sixth degree, the correspondence is one to one.

If, however, the second polar of (2) with respect to the vertex 13 be formed, f_3 is found to carry a numerical factor, thus

$$A \sim 2f_3. \tag{3}$$

41. The correspondence is very different in the case of the degenerate curve.

$$P^2 = 0, \tag{4}$$

which represents the six sides, each taken twice. The product of these at 13 is exactly f_2^2, so that

$$P^2 \sim f_2^2. \tag{5}$$

Whereas, the other 12th degree form A^2 gives the correspondence

$$A^2 \sim 2^2 f_3^2. \tag{6}$$

This illustrates theorem III [Art. 24], *showing how the factor*

$$P^{2\mu} \textit{ in the form } R_{6n}, \qquad (\mu \neq 0)$$

raises the order of the multiple point by 2μ *on account of the degeneracy of the curve* (4).

For this reason the correspondence (5) is *not one to one*, since f_2^2 may go equally well with some form of degree 18. See (10), Art. 43.

42. In order to find the binary correspondent to the curve

$$C = 0, \tag{7}$$

we form the 6th polar of (7) with reference to the point

$$13;\ 0:0:1. \tag{8}$$

For this purpose the following remarks are useful:

(a).
$$\frac{\partial p}{\partial z_i} = 1, \quad \frac{\partial q}{\partial z_i} = z_j + z_k, \quad \frac{\partial r}{\partial z_i} = z_j z_k,$$
$$(i, j, k = 1, 2, 3).$$

(b). At the point (8),
$$p = 1, \quad q = 0, \quad r = 0.$$

(c). Neither the factor p^α nor any of its derivatives below the $(\alpha + 1)^{\text{st}}$ can cause any term to vanish.

(d). The factor q^β or any of its derivatives up to and including the β^{th}, save one,
$$\frac{\partial^\beta q^\beta}{\partial z_i^\beta}, \qquad (i = 1, 2)$$
will cause all terms to vanish in which it may occur.

(e). The factor r^γ or any of its derivatives up to and including the $2\gamma^{\text{th}}$, save one,
$$\frac{\partial^{2\gamma} r^\gamma}{\partial z_1^\gamma \partial z_2^\gamma}$$
will cause all terms to vanish in which it may occur.

(f). Hence, the λ^{th} derivative of a term, $p^\alpha q^\beta r^\gamma$, will vanish for the point, $0:0:1$, unless
$$\beta + 2\gamma \leqq \lambda.$$

43. Applying these principles to the curve (9), Art. 16, we see at once that every term has
$$\beta + 2\gamma \leqq 5.$$
Hence, the first five polars must vanish. But for $\lambda = 6$ there are two terms whose sixth derivatives do not vanish,
$$-2^2 p^9 r^3 \text{ and } p^8 q^2 r^2.$$
Determining from these the sixth polar, the product of the tangential quantics at the sextuple point 13 of (7) is found to be
$$z_1^4 z_2^2 - 2 z_1^3 z_2^3 + z_1^2 z_2^4. \qquad (9)$$
This is a perfect square, and is precisely the dihedron form f_2^2. [(9), Art. 38.]

Therefore, *the curve* (7) *has three cusps at the point* 13, and hence the same also at the other three critical points.

Thus we have
$$C \sim f_2^2. \tag{10}$$

This may be called the *normal* correspondence between f_2^2 and a ternary form of degree 18, while that between f_2^2 and P^2 of degree 12 *exists only because* (4) breaks up into straight lines through the critical points.

44. It remains to discover the ternary form to which f_1^2 corresponds. From the identity (8), Art. 38, we derive
$$2f_1^2 = 2^3 f_3^3 - 2 \cdot 3^3 f_2^2. \tag{11}$$
From (3) and (10),
$$A^3 - 2 \cdot 3^3 C \sim 2^3 f_3^3 - 2 \cdot 3^3 f_2^2. \tag{12}$$
Calling the left of (12) C', we have
$$C' \sim 2f_1^2. \tag{13}$$

45. The above results may be made more general as follows :

THEOREM IV.*

If the ternary forms α, β *correspond to the binary forms* a, b *of degree* λ, μ, *then the product*
$$\alpha\beta \sim ab$$
and the sum
$$\alpha + \beta \sim a,\ b \text{ or } a + b,$$
according as
$$\lambda < \mu,\ > \mu \text{ or } = \mu.$$

The proof follows directly from the principles of higher plane curves, since the correspondence in question is determined by finding the *lowest non-vanishing polar of each curve with respect to the given point.*

By this theorem a more general form than C corresponds to f_2^2, namely,
$$C + \delta A P^2 \sim f_2^2, \tag{14}$$
where δ is an arbitrary constant.

Here the binary correspondent of C, which is of *lower degree* than that of AP^2 [Art. 39, (1)], *prevails also* for the composite form.

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* The chief theorems are numbered consecutively. See Arts. 22, 23, 24.

Likewise C' [(12), (13)] may be generalized, for if we put

$$A^3 - 2\cdot 3^3 C + \delta A P^2 = C'',$$

we have

$$C'' \sim 2f_1^2. \tag{15}$$

46. THEOREM V.

No invariant form under G_{120} can correspond to any binary form which involves odd powers of the cubic forms f_1, f_2.

Proof: If an odd power of either f_1 or f_2 alone is involved, then the form is of odd degree. This is impossible, since the degree of the *binary* form must equal the order of multiplicity of the point in question, which is always $2(n+\mu)$ according to theorem III, Art. 24.

If an odd power of the product only, $f_1 f_2$, is involved, the form is then of even degree, but is *non-symmetric* in z_1, z_2 [(10), Art. 38]. This is impossible, since the *ternary* forms are symmetric functions of z_1, z_2, z_3. [Art. 20]. Hence, all their polars with respect to the coordinate vertex, $0:0:1$, through which all the invariant curves pass [Art. 24], are *symmetric* functions in z_1, z_2, and similar statements hold with reference to the other three fundamental points.

Therefore, *all invariant forms under G_{120} have as their binary correspondents at the point** 13, *rational, integral functions of*

$$f_1^2,\ f_2^2,\ f_3.$$

47. The complete enumeration of binary correspondents for all ternary forms of degree 6, 12 and 18 may now be given as follows:

$$m_1 A \sim 2m_1 f_3,$$

$$m_1 A^2 + m_2 P^2 \sim \begin{cases} 2^2 m_1 f_3^2, & \text{if } m_1 \neq 0, \\ m_2 f_2^2, & \text{if } m_1 = 0,\ m_2 \neq 0. \end{cases}$$

$$m_1 A^3 + m_2 C + m_3 A P^2 \sim \begin{cases} m_2 f_2^2, & \text{if } m_1 = 0,\ m_2 \neq 0; \\ 2^3 m_1 f_3^3, & \text{if } m_1 \neq 0,\ m_2 = 0; \\ 2m_3 f_2^2 f_3, & \text{if } m_1 = m_2 = 0,\ m_3 \neq 0; \\ 2m_1 f_1^2, & \text{if } m_2 = -2\cdot 3^3 m_1,\ \neq 0; \\ 2^3 m_1 f_3^3 + m_2 f_2^2, & \text{if } m_2 \neq -2\cdot 3^3 m_1 \neq 0. \end{cases}$$

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*The foregoing investigation, which has been given with respect to the point 13, applies equally well to all four critical points, since these are conjugate under $G_{24}^{(1)}$.

The simplest invariant forms independent of P^2 corresponding directly to the fundamental even binary forms are

$$\left\{\begin{array}{l} \tfrac{1}{2}A \sim f_3, \\ \quad C \sim f_2^2, \\ \tfrac{1}{2}C' \sim f_1^2. \end{array}\right\} \tag{16}$$

THE GENERAL REDUCTION THEOREM. Arts. 48–52.

48. *Definition.* An invariant form is said to be *reduced* if it does not contain P^2 as a factor.

LEMMA.

Two reduced invariant forms of the same degree, which have the same binary correspondent at one of the vertices, can differ only in such terms as contain P^2 as a factor.

Proof: Let R and R' be two reduced ternary forms, each of degree $6n$, having the same tangential quantic at the point 13. We are to prove

$$R - R' = P^{2\mu} \cdot R''_{6(n-2\mu)}, \tag{1}$$

where R'' contains no factor of P and is zero if $\mu = 0$.

Since R and R' are *reduced* forms, the common tangential quantic at the point $0:0:1$, is of degree $2n$ [Art. 24], and involves only z_1 and z_2 [Art. 46]. Hence, by principles of higher plane curves,

(a). R and R' each contain no terms of degree less than $2n$ in z_1 and z_2.

(b). They are *identical* in the terms of degree $2n$ in z_1 and z_2.

(c). And, therefore, they can differ only in terms of degree *higher* than $2n$ in z_1 and z_2.

Now, $R - R'$ is an invariant form of degree $6n$, since R and R' are such by hypothesis. Then $R - R'$ must have at 13 a tangential quantic of degree $2(n+\mu)$. [Art. 24.]

Two cases now arise:

(I). $\mu = 0$. The tangential quantic is then of degree $2n$ which, by (c), is impossible, since $R - R'$ contains no terms of degree $\leqq 2n$ in z_1 and z_2. Hence,

$$R - R' = 0. \tag{2}$$

(II). $\mu \neq 0$. This means that $R - R'$ actually contains $P^{2\mu}$ as a factor, *in*

which case alone [*Art.* 41] *can a form have a tangential quantic of degree higher than* $2n$. Hence, the conclusion (I) is established.

Corollary.

49. If one of two ternary forms of the same degree has the factor $P^{2\mu}$ ($\mu \geqq 1$), while the other does not, the degree of the binary correspondent of the former is greater by 2μ than that of the latter.

Theorem VI.

50. *The most general invariant form under* G_{120} *is a rational integral function of the forms* A, P^2, C.

Proof: Let R_1 be the most general invariant form of degree $6n$, and let Q_1 be the resulting *reduced* form of degree $6(n - 2\mu_1)$.

Suppose the binary correspondent of Q_1 is given by

$$Q_1 \sim G_1(f_3,\ f_2^2,\ f_1^2), \tag{1}$$

where G_1 is a rational integral function of the *even* binary forms of degree $2(n - 2\mu_1)$ in z_1, z_2 [Arts. 24, 46.]

Now, a known invariant form of degree $6(n - 2\mu_1)$ in z_1, z_2, z_3 can be constructed, having G_1 for its binary correspondent,

$$S_1 \equiv G_1(\tfrac{1}{2}A,\ C,\ \tfrac{1}{2}C'). \tag{2}$$

For the arguments in (1) are weighted 2, 6, 6 respectively in the variables, and they are the binary forms corresponding *directly* to the arguments in (2), which are weighted 6, 18, 18 respectively. See (16), Art. 47.

Therefore, Q_1 and S_1 are two *reduced* forms having the same binary correspondent, which, by the lemma [Art. 48] are then either identical or differ only in terms involving P^2.

Thus, either

$$Q_1 - S_1 \equiv 0, \tag{3}$$

in which case the theorem is proved, or

$$Q_1 - S_1 = R_2, \tag{4}$$

where R_2 is divisible by $P^{2\mu_2}$ ($\mu_2 \geqq 1$).

Call the quotient Q_2 of degree $6(n - 2\mu_1 - 2\mu_2)$ and suppose

$$Q_2 \sim G_2(f_3, f_2^2, f_1^2). \tag{5}$$

Then, as before, set up the known form

$$S_2 \equiv G_2(\tfrac{1}{2}A, C, \tfrac{1}{2}C') \tag{6}$$

of degree $6(n - 2\mu_1 - 2\mu_2)$ and having the same binary correspondent as Q_2.

Hence, either $$Q_2 - S_2 = 0, \tag{7}$$

in which case $$R_2 = P^{2\mu_2} G_2$$

and $$R_1 = P^{2(\mu_2+\mu_1)} G_2 + P^{2\mu_1} G_1$$
$$= \phi(P^2, A, C), \tag{8}$$

or else $$Q_2 - S_2 = R_3, \tag{9}$$

where R_3 is divisible by $P^{2\mu_3}$ $(\mu_3 \geq 1)$.

If this process be continued, we must reach. after a finite number of steps $\varkappa$, a form independent of P^2, since the finite degree $6n$ is successively reduced by multiples of 6, the divisor being always a power of P^2.

Hence, *after* the $\varkappa^{\text{th}}$ reduction, we shall have

$$Q_\kappa - S_\kappa = 0 \tag{10}$$

and $$Q_\kappa = S_\kappa = G_\kappa(\tfrac{1}{2}A, C, \tfrac{1}{2}C'), \tag{11}$$

where G_κ is a function of degree $6[n - 2(\mu_1 + \mu_2 + \ldots\ldots + \mu_\kappa)]$ in z_1, z_2, z_3. So that

$$R_\kappa = P^{2\mu_\kappa} G_\kappa. \tag{12}$$

But $$Q_{\kappa-1} = R_\kappa + G_{\kappa-1} = P^{2\mu_\kappa} G_\kappa + G_{\kappa-1}.$$

Hence $$R_{\kappa-1} = P^{2(\mu_\kappa+\mu_{\kappa-1})} G_\kappa + P^{2\mu_{\kappa-1}} G_{\kappa-1}.$$

Likewise,

$$R_{\kappa-2} = P^{2(\mu_\kappa+\mu_{\kappa-1}+\mu_{\kappa-2})} G_\kappa + P^{2(\mu_{\kappa-1}+\mu_{\kappa-2})} G_{\kappa-1} + P^{2\mu_{\kappa-2}} G_{\kappa-2},$$

$$R_{\kappa-3} = P^{2(\mu_\kappa+\mu_{\kappa-1}+\mu_{\kappa-2}+\mu_{\kappa-3})} G_\kappa + P^{2(\mu_{\kappa-1}+\mu_{\kappa-2}+\mu_{\kappa-3})} G_{\kappa-1}$$
$$+ P^{2(\mu_{\kappa-2}+\mu_{\kappa-3})} G_{\kappa-2} + P^{2\mu_{\kappa-3}} G_{\kappa-3},$$

....

....

$$R_1 = P^{2(\mu_\kappa+\mu_{\kappa-1}+\ldots\ldots\mu_1)} G_\kappa + P^{2(\mu_{\kappa-1}+\mu_{\kappa-2}+\ldots\ldots\mu_1)} G_{\kappa-1} + \ldots\ldots P^{2\mu_1} G_1.$$

Therefore, we have

$$\begin{aligned} R_1 &= \phi_1(P^2, G_1, G_2, \ldots . G_\kappa) \\ &= \phi_2(P^2, A, C, C') \\ &= \phi\ (P^2. A, C), \end{aligned} \tag{13}$$

where the ϕ's are rational, integral functions.

51. Hence, *every absolute invariant form under G_{120}, throwing off the required factor in z_1, z_2, z_3, is a rational integral function of the fundamental forms A, P^2, C.*

Out of these forms must be constructed the invariant fractions, which return absolutely to themselves after canceling the common factor in numerator and denominator thrown off under any quadratic transformation of the group.

Since A is the only form of degree 6, there is no fraction of that order. The simplest fractions of degree 12, 18 and 24 respectively are

$$\frac{A^2}{P^2}\,;\ \frac{A^3}{C},\ \frac{AP^2}{C}\,;\ \frac{A^4}{P^4},\ \frac{AC}{P^4}.$$

52. It was shown [Art. 17] that the form P is the only *fundamental relative invariant form under $G_{24}^{(1)}$*, and that it throws off the factor (-1). It also throws off (-1) under the generator T' in addition to the factor r^3 in the z's required by the theorem of Art. 22.

It is, therefore, the *fundamental relative invariant form under G_{120}*, and all other relative invariants are formed from the product

$$P^{2\kappa+1}\cdot\phi\,(A,\ P^2,\ C).$$

However, P is an absolute invariant under the alternating group G_{60}, and the complete form-system for this subgroup is, therefore,

$$A,\ P,\ C.$$

There is then for G_{60} an absolute invariant fraction of degree 6,

$$\frac{A}{P}.$$

There is no form of degree 12 in addition to those given in Art. 51

For the degrees 18 and 24 the following non-reducible forms for G_{60} are to be added:

$$\frac{A^3}{P^3}, \quad \frac{C}{P^3}, \quad \frac{A^2P}{C}; \quad \frac{A^4}{PC}.$$

It will be seen that these fractions are all of type (III) or (IV), Art. 19, in which forms all relative invariant fractions under G_{120} must occur.

THE UNIVERSITY OF CHICAGO, *Dec.* 1, 1900.

NOTE.—In Part First, the last foot-note to Art. 1 should read, pp. 279–291, and the last foot-note to Art. 3 should read, p. 283. The heading above Art. 17 should read, Arts. 17–20.

Memoir on the Algebra of Symbolic Logic.

By A. N. Whitehead, *Fellow of Trinity College, Cambridge, England.*

Preface.

The present memoir is a purely mathematical investigation concerning the Algebra of Symbolic Logic. As a matter of history, this algebra has only been continuously studied since the publication of Boole's "Laws of Thought" (1854), and to C. S. Peirce* and to Schröder,† must be assigned the credit of perfecting its laws of operation. But, as a question of logical priority, this algebra must be considered as the first object of mathematical study. It holds this position by a double right.

First, its interpretation is concerned with the fundamental conceptions of classes, and of their mutual inclusions and exclusions; thus the terms of the algebra, such as a, b, c, ..., x, y, z, can be interpreted as each representing a class; a sum of terms, such as $a + b$, represents the class formed by the two classes a and b; the product of terms, such as ab, represents the class common to the two classes a and b; the symbol i represents all the classes which are the subject of discourse, so that not-a is conceived as a class composed of all i with the exception of a, and is denoted by $\bar{a}$; the symbol 0 means nonentity.

Secondly, the symbolism of the algebra is the simplest of all algebraic systems. It is explained in §1, Part I, of this memoir. The laws $a + a = a$, $aa = a$, enable the use of numerals to be avoided, either as factors or indices. Also, the associative, commutative and distributive laws are preserved.

But the algebra has apparently had the defects natural to its simplicity. It is like argon in relation to the other chemical elements, inert and without

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* Proc. of Amer. Acad. of Arts and Sciences, 1867, and Amer. Journ. of Math., vols. III and IV.

† "Operationskreis des Logikkalküls," 1877, "Vorlesungen über die Algebra des Logik," vol. I, 1890, vol. II, 1891.

intrinsic activities. Accordingly, the purely mathematical study of the algebra has languished; the theory of Duality, the theory of Development by the process of Dichotomy, and the theory of Equations, have apparently exhausted its purely mathematical properties. The investigations concerning these properties were commenced by Boole, and have been brought to a high degree of perfection by Schröder. But the greater part of the literature relating to this algebra may be termed "applied mathematics;"* it is primarily concerned with the relation of the symbolism to its interpretation in the field of Logic,† and with its use as a practical means for the exact expression of deductive reasoning, especially in regard to the foundations of the various branches of mathematics.‡

In the present memoir, it is shown that the algebra has many more purely mathematical properties than those with which it has hitherto been credited. In the theories of ordinary algebraic symbols, the solution of equations is not the dominant subject of enquiry. Analogously, here the centre of interest has been shifted from the solution of equations to the study of the properties of functions of independent variables.

The keynote of this memoir is the prominence given to three ideas, namely, that of the "invariants" of a function of independent variables, that of "prime functions of independent variables," and that of the theory of "substitutions" of independent variables for independent variables. This last idea connects the algebra with the theory of Groups, and opens out a large field for investigation in that direction.

Invariants are defined in §2 of Part I: In the course of the memoir, the leading properties of functions of independent variables are shown to depend on them. Primes are defined in §3 of Part I. The expression here called a "primary prime" has been noticed by Jevons, and some of its elementary properties are discussed by Schröder§ and Peano.|| But here a different use is made

* Though the field of the application is itself "pure."

† Cf. Venn's "Symbolic Logic," 1st ed., 1881; 2d ed., 1894.

‡ Cf. the admirable work in this direction of the Italian school, originated and inspired by Peano. The following are some of its principal works: "Arithmetices Principiæ," by Peano, Turin, 1889; "Notations de Logique Mathématique," by Peano, Turin, 1894; "Formulaire de Mathématiques," Tome I, Turin, 1895; ibid., Tome II, No. 1, 1897; ibid., No. 2, 1898; ibid., No. 3, 1899; "Revue de Mathématiques," Turin, Tome VII, No. 1; ibid., passim.

§ Cf. loc. cit., vol. I, "Neunte Vorlesung," pp. 370, 371; pp. 380, 381.

|| Cf. "Formulaire de Math.," 1895, I, §3, propositions 24 to 30.

of them, and from them are constructed other functions called "n-ary linear primes" and "n-ary separable primes."

In §5 of Part I, the properties of primes in relation to factorization and summation are developed; and the fundamental nature of primes is here made evident.

In §6, the factorization of any evanescible function of n variables into n-ary linear primes is discussed. An evanescible function is a function which can be made to vanish by an appropriate choice of values for its variables. This factorization of a given evanescible function into n-ary linear primes can be carried out in an indefinite number of ways. But the fundamental theorem is proved that, in general, the minimum number of such factors is 2^n. An exceptional evanescible function for which the minimum number is $2^n - r$ is said to be of deficiency r. The deficiency is shown to depend on the vanishing of r of the invariants. The term "deficiency" had previously been formally defined in §2 in relation to these invariants. The term has also a meaning, arising out of its present meaning, in regard to non-evanescible functions (cf. §8).

In §7, the theorems deduced from §6 by the method of duality are enunciated. These relate to the expression of functions of n variables as sums of separable primes. The property, analogous by the theory of duality to deficiency, is called "supplemental deficiency."

In §9, two special types of functions called "linear" and "separable" functions are discussed, and various theorems relating to them are proved.

Part II is devoted to the theory of "Substitutions." A function $\phi(x, y)$ is transformed into some function $\psi(u, v)$ by the transformation $x = f_1(u, v)$, $y = f_2(u, v)$. But, in general, it is not allowable to conceive x and y as independent variables, when this transformation is used. For the condition of the possibility of the two equations of transformation, viewed as equations to find u and v, is an equation which x and y must satisfy. Thus x and y are restricted to be simultaneous solutions of this equation. But this equation reduces to the identity, $0 = 0$, for all values of x and y, if the coefficients of the functions $f_1(u, v)$ and $f_2(u, v)$ satisfy a certain condition. In this case, the transformation amounts to the substitution of one set of independent variables for another set of independent variables. The term "substitution" is exclusively used for this type of transformation. Each substitution is denoted by a single letter; also u and v are replaced by x and y. Thus we write $Tx = f_1(x, y)$, $Ty = f_2(x, y)$; also $T\phi(x, y)$ is written for $\phi(Tx, Ty)$. These explanations occupy §1 of Part II.

In §2, the relations between the coefficients of $f_1(x, y)$ and $f_2(x, y)$, that is, of Tx and Ty, are more fully discussed. It is proved that both Tx and Ty are of deficiency two and of supplemental deficiency two; and the two functions are also otherwise related.

In §3, it is proved that there is only one reverse substitution corresponding to a given substitution T, and its coefficients are determined. It follows (cf. §4) that all possible substitutions form a group. The group is not continuous, since the concepts of "real number" and of infinitesimal variations of real numbers have no place in this algebra. It is of finite order, if the number of distinct fundamental terms in the algebra, representing constants, is conceived as finite. It is of indefinite order in so far as these fundamental given constant terms are not brought into explicit definition; and also in so far as new fundamental constants may always be produced at discretion without violating any law of the algebra. The order of the subgroup T, T^2, T^3, ... is evidently finite and determinable in the general case.*

In §6, the "congruence" of functions is defined. Two functions, $\phi(x, y)$ and $\Phi(x, y)$, are defined to be "congruent," if any substitution T exists such that $T\phi(x, y) = \Phi(x, y)$. The fundamental theorem is proved that all functions are congruent which have all their corresponding invariants equal. A complete set of congruent functions is called a congruent family.

The subgroup of substitutions (cf. §7) which leave a given function unchanged is called the identical group of the function. It is proved that the identical groups of all members of the same congruent family are simply isomorphic. It is also proved (cf. §8) that, except in assigned special cases, the subgroup common to the identical groups of any two functions, contains more than the single identical substitution T^0.

PART I.

THE THEORY OF PRIMES.

§1.—*Elementary Principles.*

The following summary of the elements of the algebra may be useful:

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*I have since determined the order of this group, which is, in general, of the 12th order, and the orders of all other groups mentioned in this memoir. I hope shortly to publish these results, which depend on a new general method in connection with this subject.—*Note added February*, 1901.

Let a, b, c be any terms subject to the laws of the algebra, then

$$a+b=b+a, \quad a+b+c=(a+b)+c=a+(b+c).$$
$$a+a=a, \quad a+0=a, \quad a+i=i.$$
$$ab=ba, \quad abc=ab.c=a.bc.$$
$$aa=a, \quad a0=0a=0, \quad ai=ia=a.$$

The supplement of any term a is written $\bar{a}$, and is defined by

$$a+\bar{a}=i, \quad a\bar{a}=0.$$

The supplement of a complex expression, such as $(a+b)$, is written $\overline{}(a+b)$ We have

$$\overline{}(a+b)=\bar{a}\bar{b}, \quad \overline{}(ab)=\bar{a}+\bar{b}.$$

Also $\bar{\bar{a}}=a, \quad \bar{0}=i, \quad \bar{i}=0.$

The equation $P+Q=0$

implies $P=0$, $Q=0$, and conversely. Thus, any number of equations in which the right-hand sides are zero, can be combined into one equation by simple addition of their left-hand sides.

The equation $P=Q$ is equivalent to

$$P\bar{Q}+Q\bar{P}=0.$$

Thus any equation can be transformed into one with its right-hand side zero.

The equation $P=Q$ implies $\bar{P}=\bar{Q}$, and conversely. The general "developed" form of a function of n independent variables x, y, ... t is

$$Axy \ldots\ldots t+Bxy \ldots\ldots \bar{t}+\ldots\ldots+K\bar{x}\bar{y} \ldots\ldots \bar{t},$$

where every possible product involving x or $\bar{x}$, y or $\bar{y}$, ..., t or $\bar{t}$, is represented, and the coefficients represent constants.

The supplement of this function is

$$\bar{A}xy \ldots\ldots t+\bar{B}xy \ldots\ldots \bar{t}+\ldots\ldots+\bar{K}\bar{x}\bar{y} \ldots\ldots \bar{t}.$$

For the sake of brevity, the argument will be conducted with respect to functions of two variables; but the leading theorems can easily be generalized for functions of any number of variables. The typical form of a function, $\phi(x, y)$, of two variables will always be written

$$Axy+Bx\bar{y}+C\bar{x}y+D\bar{x}\bar{y},$$

and this notation for the coefficients will always be adhered to. Then

$$\phi(x, y) = Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}.$$

The condition that a proposed equation in n variables, $x, y, \ldots . t$, such as

$$\psi(x, y, \ldots . t) = 0,$$

may be a possible equation, is found by substituting i or 0 in $\psi(x, y, \ldots . t)$ for each of $x, y, \ldots . t$ in every possible combination, and by equating to zero the product of the results of such substitutions. This equation of condition will be symbolically represented by

$$\prod \psi \begin{pmatrix} i, & i, & \ldots . & i \\ 0, & 0, & \ldots . & 0 \end{pmatrix} = 0.$$

Thus the condition for the possibility of

$$\phi(x, y) = 0$$

is

$$ABCD = 0,$$

and the condition for the possibility of

$$Hx + K\bar{x} = 0$$

is

$$HK = 0.$$

The general solution of this latter equation is

$$x = K + u\bar{H},$$

where u is an arbitrary unknown. Thus since an indefinite number of special values can be given to u, every equation has, in general, an indefinite number of particular roots which are all included in the general solution.

According to the duality,* to every symbolic theorem involving $+$, $\times$ (the symbol for multiplication, usually omitted), i, 0, there corresponds a symbolic theorem in which $+$ is replaced by $\times$, $\times$ by $+$, i by 0, 0 by i. An outcome of this theory in its application to functions of independent variables is as follows: From the theorem that $\phi(x, y)$ can be expressed in the form $\psi(x, y)$, when the condition $\chi(A, B, C, D) = 0$ is fulfilled by the coefficients of $\phi(x, y)$, there can be derived the theorem that $\phi(x, y)$ can be expressed in the form $\bar{\psi}(x, y)$ when the condition $\chi(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = 0$ is fulfilled.

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*Cf. my "Universal Algebra," §24, for a full statement and proof of this theory. The theory is due to C. S. Peirce and to Schröder.

§2.—*Symmetric Functions.*

Consider n terms $a_1, a_2, \ldots . a_n$. Let their symmetric functions be defined as follows, by a notation which will be adhered to:

$$S_1 = \sum_{p=1}^{n} a_p, \quad S_2 = \sum_{p,\, q=1}^{n} a_p a_q, \quad S_3 = \sum_{p,\, q,\, r=1}^{n} a_p a_q a_r, \; \ldots . , \; S_n = a_1 a_2 \ldots . a_n$$

where, in S_2, it is understood that the subscript p is not equal to q in the same product $a_p a_q$, and similarly for $S_3, S_4, \ldots . S_n$. This, or a similar supposition respecting the inequality of suffixes when typical products or sums are given, will be adhered to unless it is otherwise expressly stated. Then

$$\bar{S}_n = \sum_{p=1}^{n} \bar{a}_p, \quad \bar{S}_{n-1} = \sum_{p,\, q=1}^{n} \bar{a}_p \bar{a}_q, \quad \bar{S}_{n-2} = \sum_{p,\, q,\, r=1}^{n} \bar{a}_p \bar{a}_q \bar{a}_r, \; \ldots . \; \bar{S}_1 = \bar{a}_1 \bar{a}_2 \ldots . \bar{a}_n .$$

Also, evidently,

$$S_1 \mathrel{=\!\!\!\!\ni} S_2 \mathrel{=\!\!\!\!\ni} S_3 \ldots . \mathrel{=\!\!\!\!\ni} S_n,$$
$$\bar{S}_1 \mathrel{=\!\!\!\!\in} \bar{S}_2 \mathrel{=\!\!\!\!\in} \bar{S}_3 \ldots . \mathrel{=\!\!\!\!\in} \bar{S}_n .$$

These subsumptions can all be expressed in the typical equation

$$S_{p+q} \bar{S}_p = 0. \tag{1}$$

This equation is also equivalent to any one of the following forms:

$$S_p S_{p+q} = S_{p+q}, \quad S_p + S_{p+q} = S_p, \quad \bar{S}_p \bar{S}_{p+q} = \bar{S}_p, \quad \bar{S}_p + \bar{S}_{p+q} = \bar{S}_{p+q}. \tag{2}$$

All other symmetric functions of $a_1, a_2, \ldots . a_n$, and of their supplements, can be expressed in terms of these fundamental symmetric functions. Thus

$$\left.\begin{array}{l} S_1 \bar{S}_n = \Sigma a_p \bar{a}_q, \quad S_1 \bar{S}_{n-1} = \Sigma a_p \bar{a}_q \bar{a}_r, \quad S_1 \bar{S}_{n-2} = \Sigma a_p \bar{a}_q \bar{a}_r \bar{a}_s, \ldots . \\ S_2 \bar{S}_n = \Sigma a_p \bar{a}_q \bar{a}_r, \quad S_2 S_{n-1} = \Sigma a_p a_q \bar{a}_r \bar{a}_s, \ldots . \end{array}\right\} \tag{3}$$

It follows from equations (2) that if $S_p = 0$, then $S_{p+1}, S_{p+2}, \ldots . S_n$ all vanish, and that if $\bar{S}_p = 0$, then $\bar{S}_{p-1}, \bar{S}_{p-2}, \ldots . \bar{S}_1$ all vanish.

The symmetric functions of the coefficients of the function $\phi(x, y, \ldots . t)$ will be called the invariants of the function. Thus, for the function $\phi(x, y)$ of two variables, the invariants are

$$\left.\begin{array}{l} S_1 = A + B + C + D, \quad S_2 = AB + AC + AD + BC + BD + CD, \\ S_3 = ABC + ABD + ACD + BCD, \quad S_4 = ABCD, \\ \bar{S}_1 = \bar{A}\bar{B}\bar{C}\bar{D}, \quad \bar{S}_2 = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}\bar{D} + \bar{A}\bar{C}\bar{D} + \bar{B}\bar{C}\bar{D}, \\ \bar{S}_3 = \bar{A}\bar{B} + \bar{A}\bar{C} + \bar{A}\bar{D} + \bar{B}\bar{C} + \bar{B}\bar{D} + \bar{C}\bar{D}, \quad \bar{S}_4 = \bar{A} + \bar{B} + \bar{C} + \bar{D}, \end{array}\right\} \tag{4}$$

If the condition for any special property of a function can be expressed as a relation between its invariants, without the coefficients otherwise entering into the relation, the property will be called an invariant property. Such invariant properties will be proved in Part II to be analogous in many respects to invariant properties of rational integral algebraic forms in ordinary algebra.

The following definitions of technical terms may be stated at once, though their fitness will only be apparent when primes have been introduced. Let $\phi(x, y, \ldots . t)$ be a function of n variables, and let $S_1, S_2, \ldots . S_{2^n}$ be its 2^n invariants. Then if $\overline{S_1} = 0$, $\phi(x, y, \ldots . t)$ will be said to be of "deficiency one" at least; if $\overline{S_2} = 0$, the function will be said to be of "deficiency two" at least, and so on. Thus, if $\overline{S_p} = 0$, the function is of "deficiency p" at least.

Again, if $S_{2^n} = 0$, $\phi(x, y, \ldots . t)$ will be said to be of "supplemental deficiency one" at least; if $S_{2^n - 1} = 0$, of "supplemental deficiency two" at least, and if $S_{2^n - p} = 0$, of "supplemental deficiency $p + 1$" at least.

Now, if $S_{2^n} = 0$, the equation

$$\phi(x, y, \ldots . t) = 0$$

is a possible equation. In this case the function will also be called evanescible. Thus an evanescible function is of supplemental deficiency one, and conversely.

If $\overline{S_1} = 0$, then the equation

$$\overline{\phi}(x, y, \ldots . t) = 0, \text{ that is, } \phi(x, y, \ldots . t) = i$$

is a possible equation. In this case, the function will be said to be "capable of the value i." Thus a function capable of the value i is of deficiency one, and conversely.

§3.—*Primes*

A function of one variable x, which is of the special type

$$\overline{a}x + a\overline{x},$$

where a is any constant, will be called a "primary prime." It will be written for brevity in the form $p(a, x)$. It is obvious that $p(a, x) = p(x, a)$. Then

$$\overline{p}(a, x) = ax + \overline{a}\overline{x} = p(\overline{a}, x).$$

Thus the supplement of a primary prime is itself a primary prime.

The solution of the equation

$$p(a, x) = 0$$

is

$$x = a + ua = a.$$

Thus the solution is definite, involving no arbitrary unknown. In other words, the equation has only one root.

Conversely, if the equation

$$Hx + K\bar{x} = 0$$

has only one root, then the function $Hx + K\bar{x}$ is a primary prime. For the solution is

$$x = K + u\bar{H}.$$

Hence, since by hypothesis, the equation has only one root,

$$\bar{H} \nleqq K.$$

But, since the equation is possible,

$$HK = 0.$$

that is,

$$K \nleqq \bar{H}.$$

Hence,

$$\bar{H} = K.$$

Thus

$$Hx + K\bar{x} = Hx + \bar{H}\bar{x} = p(\bar{H}, x).$$

A function of the n variables $x, y, z, \ldots.$ which can be expressed in the form

$$\bar{a}x + a\bar{x} + \bar{b}y + b\bar{y} + \bar{c}z + c\bar{z} + \ldots.$$

will be called an "n-ary linear prime." It can be written

$$p(a, x) + p(b, y) + p(c, z) + \ldots. ;$$

and for brevity it will be written

$$p(a, x;\ b, y;\ c, z;\ \ldots.).$$

The supplement of an n-ary linear prime is

$$\begin{aligned}\bar{p}(a, x;\ b, y;\ c, z;\ \ldots.) &= \bar{p}(a, x)\,\bar{p}(b, y)\,\bar{p}(c, z)\ \ldots. \\ &= p(\bar{a}, x)\,p(\bar{b}, y)\,p(\bar{c}, z)\ \ldots. .\end{aligned}$$

This type of function of the n variables $x, y, z, \ldots.$ will be called an "n-ary separable prime." Thus for two variables x and y, we have secondary linear primes, symbolized thus:

$$p(a, x;\ b, y) = p(a, x) + p(b, y) = ax + a\bar{x} \bar{+} by + by;$$

and secondary separable primes, symbolized thus:

$$\bar{p}\,(a,\,x;\;b,\,y) = p\,(\bar{a},\,x)\,p\,(\bar{b},\,y) = (ax + \bar{a}\bar{x})(by + \bar{b}\bar{y}).$$

A primary prime can be looked on both as a linear prime of one variable, and as a separable prime of one variable.

It must be noted that an n-ary prime, either linear or separable, is not a degenerate form of a $(n+m)$-ary prime of the same type. Thus, $\bar{a}x + a\bar{x}$ is not a degenerate form of $\bar{a}x + a\bar{x} + \bar{b}y + b\bar{y}$; for no constant value of b can be found which will make $\bar{b}y + b\bar{y}$ vanish for all values of y. Similarly, $\bar{a}x + a\bar{x}$ is not a degenerate form of $(\bar{a}x + a\bar{x})(\bar{b}y + b\bar{y})$.

It is one of the leading objects of Part I of this memoir to prove that primes are to be considered as the fundamental types of function of this algebra, and that all functions of unknowns can be conveniently classified according as they are constructed of primes.

The condition that $\phi\,(x,\,y)$ may be a secondary linear prime is found by identifying it with $p\,(a,\,x;\;b,\,y)$. But we may write

$$p\,(a,\,x;\;by) = (\bar{a} + b)\,xy + (\bar{a} + b)\,x\bar{y} + (a + \bar{b})\,\bar{x}y + (a + b)\,\bar{x}\bar{y}.$$

Hence, by comparison of coefficients,

$$A = \bar{a} + \bar{b},\quad B = \bar{a} + b,\quad C = a + \bar{b},\quad D = a + b.$$

These equations can be written as the single equation

$$(A + \bar{B} + \bar{C} + \bar{D})\,ab + (\bar{A} + B + \bar{C} + \bar{D})\,a\bar{b} + (\bar{A} + \bar{B} + C + \bar{D})\,\bar{a}b \\ + (\bar{A} + \bar{B} + \bar{C} + D)\,\bar{a}\bar{b} = 0$$

The resultant of this equation for a and b is

$$(A + \bar{B} + \bar{C} + \bar{D})(\bar{A} + B + \bar{C} + \bar{D})(\bar{A} + \bar{B} + C + \bar{D})(\bar{A} + \bar{B} + \bar{C} + D) = 0.$$

Hence, after multiplying out and reducing, we find

$$S_4 + \bar{S}_3 = 0; \tag{4}$$

which is the necessary and sufficient condition for a secondary linear prime.

Thus a secondary linear prime is evanescible and capable of the value i. Its invariants are $S_1 = i$, $S_2 = i$, $S_3 = i$, $S_4 = 0$. It is of deficiency three and of supplemental deficiency one.

Reciprocally, we can state without further proof, that the necessary and sufficient condition that $\phi(x, y)$ may be a secondary separable prime is

$$\bar{S}_1 + S_2 = 0. \tag{5}$$

Thus, a secondary separable prime is capable of the value i and is evanescible. Its invariants are $S_1 = i$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$. It is of deficiency one, and of supplemental deficiency three. It is evident that the same function cannot be both a secondary linear prime and a secondary separable prime.

It can be proved that a general form for a secondary linear prime is

$$(p + q + r)\,xy + (\bar{p} + q + r)\,x\bar{y} + (\bar{q} + r)\,\bar{x}y + \bar{r}\bar{x}\bar{y}; \tag{6}$$

and that a general form for a secondary separable prime is

$$pqrxy + \bar{p}qrx\bar{y} + \bar{q}r\bar{x}y + \bar{r}\bar{x}\bar{y}. \tag{7}$$

Another form for the condition (4), that $\phi(x, y)$ may be a secondary linear prime, can be proved to be

$$\bar{A} = BCD,\quad \bar{B} = ACD,\quad \bar{C} = ABD,\quad \bar{D} = ABC. \tag{8}$$

Reciprocally, another form for the condition (5), that $\phi(x, y)$ may be a secondary separable prime is

$$A = \bar{B}\bar{C}\bar{D},\quad B = \bar{A}\bar{C}\bar{D},\quad C = \bar{A}\bar{B}\bar{D},\quad D = \bar{A}\bar{B}\bar{C}. \tag{9}$$

The solution of the equation

$$p(a, x;\ b, y) = 0$$

is $x = a$, $y = b$. Thus the equation has only one set of roots.

Conversely, if the equation

$$\phi(x, y) = 0$$

has only one set of roots, then $\phi(x, y)$ is a secondary linear prime.

For, by eliminating y, we find

$$ABx + CD\bar{x} = 0.$$

This must have only one root, and accordingly

$$AB = \bar{C} + \bar{D}.$$

Similarly, in order that the equation for x may only have one root, we must have

$$AC = \overline{B} + \overline{D}.$$

These equations are equivalent to the single equation

$$ABCD + (\overline{A} + \overline{B})(\overline{C} + \overline{D}) + (\overline{A} + \overline{C})(\overline{B} + \overline{D}) = 0,$$

that is,

$$S_4 + \overline{S}_3 = 0.$$

Hence, $\phi(x, y)$ is a secondary linear prime. A similar theorem holds for equations involving any number of unknowns.

The condition that $\phi(x, y)$ is a primary prime with x as sole variable, is evidently

$$\overline{A} = \overline{B} = C = D. \tag{10}$$

This is not an invariant condition, but it involves

$$S_1 = i, \quad S_2 = i, \quad S_3 = 0, \quad S_4 = 0.$$

Thus a primary prime $p(a, x)$, if it is written in the form $(y + \overline{y})\, p(a, x)$ so as to appear as a function of two variables, x and y, is a function of deficiency two and of supplemental deficiency two.

§4.—*Factorization and Expression as Sums.*

Any function can be factorized in an indefinite number of ways. For assume

$$\phi(x) = Hx + K\overline{x} = (H_1 x + K_1 \overline{x})(H_2 x + K_2 \overline{x}) = H_1 H_2 x + K_1 K_2 \overline{x}.$$

Then

$$H = H_1 H_2, \quad K = K_1 K_2.$$

Hence,

$$\overline{H}_1 H = 0, \qquad H_1 = H + p_1.$$

Also

$$H_2 = H + p_2 \overline{H}_1 = H + \overline{p}_1 p_2 \overline{H} = H + \overline{p}_1 p_2.$$

Similarly,

$$K_1 = K + q_1, \quad K_2 = K + \overline{q}_1 q_2.$$

Thus,

$$\phi(x) = Hx + K\bar{x} = \{(H + p_1)x + (K + q_1)\bar{x}\}\{(H + \bar{p}_1 p_2)x + (K + \bar{q}_1 q_2)\bar{x}\}. \quad (11)$$

Similarly,

$$\phi(x, y) = \{A + p_1)xy + (B + q_1)x\bar{y} + (C + r_1)\bar{x}y + (D + s_1)\bar{x}\bar{y}\}$$
$$\times \{(A + \bar{p}_1 p_2)xy + (B + \bar{q}_1 q_2)x\bar{y} + (C + \bar{r}_1 r_2)\bar{x}y + (D + \bar{s}_1 s_2)\bar{x}\bar{y}\}. \quad (12)$$

Thus the most general types of pairs of factors have been found. They can be expressed otherwise thus: Let $\psi_1(x, y)$ and $\psi_2(x, y)$ be any two functions of x and y, then the most general type of fact or of $\phi(x, y)$ is

$$\phi(x, y) + \psi_1(x, y);$$

and the most general type of a pair of factors is expressed by

$$\phi(x, y) = \{\phi(x, y) + \psi_1(x, y)\}\{\phi(x, y) + \bar{\psi}_1(x, y)\psi_2(x, y)\}. \quad (13)$$

Equation (13) is evidently identical with equation (12) when we put

$$\psi_1(x, y) = p_1 xy + q_1 x\bar{y} + r_1 \bar{x}y + s_1 \bar{x}\bar{y},$$
$$\psi_2(x, y) = p_2 xy + q_2 x\bar{y} + r_2 \bar{x}y + s_2 \bar{x}\bar{y}.$$

Reciprocally, any function $\phi(x, y)$ can be expressed as the sum of two functions in an indefinite number of ways; thus, if we put

$$\phi(x, y) = \phi_1(x, y) + \phi_2(x, y),$$

the most general type for $\phi_1(x, y)$ is

$$\phi_1(x, y) = \phi(x, y)\psi_1(x, y); \quad (14)$$

and if $\phi_1(x, y)$ be given by (14), the most general type for $\phi_2(x, y)$ is

$$\phi_2(x, y) = \phi(x, y)\{\bar{\psi}_1(x, y) + \psi_2(x, y)\}. \quad (15)$$

Let $\phi_1(x, y)$ and $\phi_2(x, y)$ be called "summands" of $\phi(x, y)$, the word "summand" corresponding to the word "factor" in relation to products.

We have thus seen that any function $\phi(x, y)$ can be factorized, or expressed as a sum, in an indefinite number of ways. It remains to consider the special conditions which factors or summands can be made to obey.

It is evident from (12) that a factor of $\phi(x, y)$ cannot have supplemental deficiency of higher order than the supplemental deficiency of $\phi(x, y)$. For

$$(A + p_1)(B + q_1) = 0,$$

implies $$AB = 0;$$
and $$(A + p_1)(B + q_1)(C + r_1) = 0,$$
implies $$ABC = 0;$$
and so on.

Again, from (12), it follows that a factor of $\phi(x, y)$ must have deficiency of at least equal order to that of $\phi(x, y)$, and may have deficiency of a higher order.

For if $$\bar{A}\bar{B} = 0,$$
it follows that
$$\overline{}(A + p_1)\,\overline{}(B + q_1) = \bar{A}\bar{B}\bar{p}_1\bar{q}_1 = 0;$$
and if $$\bar{A}\bar{B}\bar{C} = 0,$$
it follows that
$$\overline{}(A + p_1)\,\overline{}(B + q_1)\,\overline{}(C + r_1) = \bar{A}\bar{B}\bar{C}\bar{p}_1\bar{q}_1\bar{r}_1 = 0;$$
and so on.

Reciprocally, a summand of $\phi(x, y)$ cannot have deficiency of higher order than the deficiency of $\phi(x, y)$, and it must have supplemental deficiency of at least equal order to that of $\phi(x, y)$.

For example, no function can have an evanescible factor unless it be itself evanescible; and every factor of a function capable of the value i is itself capable of the value i. Reciprocally, no function can have a summand capable of the value i unless it be itself capable of the value i; and every summand of an evanescible function is itself evanescible.

If $\phi(x, y)$ be evanescible, then an evanescible companion factor to any possible factor can always be found. For any possible factor can be written in the form
$$(A + p_1)\,xy + (B + q_1)\,x\bar{y} + (C + r_1)\,\bar{x}y + (D + s_1)\,\bar{x}\bar{y}.$$

The most general type of its companion factor is
$$(A + \bar{p}_1 p_2)\,xy + (B + \bar{q}_1 q_2)\,x\bar{y} + (C + \bar{r}_1 r_2)\,\bar{x}y + (D + \bar{s}_1 s_2)\,\bar{x}\bar{y}.$$

This is evanescible if
$$(A + \bar{p}_1 p_2)(B + \bar{q}_1 q_2)(C + \bar{r}_1 r_2)(D + \bar{s}_1 s_2) = 0.$$

But this is always a possible equation for p_2, q_2, r_2, s_2, since, by hypothesis, one set of roots is $p_2 = q_2 = r_2 = s_2 = 0$.

§5.—*Properties of Primes.*

A linear prime has no evanescible factor other than itself. And conversely, if an evanescible function has no evanescible factor other than itself, it is a linear prime. First consider the primary prime $p(a, x)$. Any factor must be of the form $(\bar a + p)\, x + (a + q)\, \bar x$. This factor is evanescible if

$$(\bar a + p)(a + q) = 0,$$

that is, if

$$ap + \bar a q + pq = 0.$$

But $ap = 0$ implies $p = u\bar a$, and $\bar a q = 0$ implies $q = va$. Thus

$$(\bar a + p)\, x + (a + q)\, \bar x = (\bar a + u\bar a)\, x + (a + va)\, \bar x = \bar a x + a\bar x.$$

Hence, the evanescible factor is merely the original primary prime.

Again, consider the secondary linear prime $p(a, x;\ b, y)$. Any factor of it must be of the form

$$(\bar a + \bar b + p)\, xy + (\bar a + b + q)\, x\bar y + (\bar a + b + r)\, \bar x y + (a + b + s)\, xy.$$

This factor is evanescible if

$$(\bar a + \bar b + p)(\bar a + b + q)(a + \bar b + r)(a + b + s) = 0.$$

Hence, $\quad p\,(\bar a + b)(a + \bar b)(a + b) = 0,$

that is, $\quad pab = 0.$

Hence, $\quad p = u\,(\bar a + \bar b).$

Similarly, $\quad q = u_2\,(\bar a + b),\quad r = u_3\,(a + \bar b),\quad s = u_4\,(a + b).$

Thus the evanescible factor reduces to the original secondary linear prime. A similar proof holds for any n-ary linear prime.

Conversely, let $\phi(x, y)$ be an evanescible function which has no evanescible factor other than itself. Then, by hypothesis,

$$ABCD = 0. \qquad \text{(a)}$$

Also any factor is of the form

$$(A + p)\, xy + (B + q)\, x\bar y + (C + r)\, \bar x y + (D + s)\, \bar x\bar y.$$

And by hypothesis, if this factor is evanescible, it must reduce to the function $\phi(x, y)$. Hence, assuming evanescibility,

$$p = u_1 A, \quad q = u_2 B, \quad r = u_3 C, \quad s = u_4 D;$$

that is, $$p\bar{A} = 0, \quad q\bar{B} = 0, \quad r\bar{C} = 0, \quad s\bar{D} = 0. \tag{b}$$

But the condition for evanescibility is

$$(A + p)(B + q)(C + r)(D + s) = 0.$$

Eliminating q, r, s, we find

$$(A + p)\, BCD = 0;$$

hence, $$p = ABCD + v(\bar{B} + \bar{C} + \bar{D}) = v(\bar{B} + \bar{C} + \bar{D}),$$

by the use of equation (a).

But the first of equations (b) is to hold for every value of p which is consistent with the evanescibility of the factor. Hence,

$$v(\bar{B} + \bar{C} + \bar{D})\,\bar{A} = 0,$$

for every value of v.

Hence, $$\bar{A}\bar{B} + \bar{A}\bar{C} + \bar{A}\bar{D} = 0.$$

Similarly for q, r, s. Thus we find

$$\bar{S}_3 = 0.$$

By combining with equation (a), we deduce

$$S_4 + \bar{S}_3 = 0.$$

This is the condition that $\phi(x, y)$ may be a secondary linear prime. A similar proof of this converse part of the theorem holds for a function of any number of variables. Reciprocally, a separable prime has no summand capable of the value i other than itself. And conversely, if a function, capable of the value i has no summand capable of the value i, other than itself, it is a separable prime.

A linear prime cannot be expressed as the product of two factors, of which one is constant and other than i.

For assume

$$p(a, x; b, y) = d(A_1 xy + B_1 x\bar{y} + C_1 \bar{x}y + D_1 \bar{x}\bar{y}).$$

Then $\quad dA_1 = \bar{a} + \bar{b}, \quad dB_1 = \bar{a} + b, \quad dC_1 = a + \bar{b}, \quad dD_1 = a + b.$

Thus
$$d(A_1 + B_1 + C_1 + D_1) = i.$$
Hence,
$$d = i.$$

Reciprocally, a separable prime cannot be expressed as the sum of two summands, of which one is constant and other than 0.

The sum of two distinct linear primes, functions of the same variables, is not evanescible.

For if $p(a_1, x) + p(a_2, x)$ is evanescible, then
$$(\bar{a}_1 + \bar{a}_2)(a_1 + a_2) = 0,$$
that is,
$$\bar{a}_1 a_2 + a_1 \bar{a}_2 = 0.$$
Hence,
$$a_1 = a_2.$$
Again, consider $p(a_1, x;\ b_1, y) + p(a_2, x;\ b_2, y)$. It is evident that this is only evanescible if
$$p(a_1, x) + p(a_2, x) \text{ and } p(b_1, y) + p(b_2, y)$$
are both evanescible. But this requires
$$a_1 = a_2 \text{ and } b_1 = b_2.$$

Reciprocally, the product of two distinct separable primes, functions of the same variables, is not capable of the value i.

The properties of primes proved in this section are the reason for the name "prime" here assigned to them: and they are also the foundation of the importance of primes, linear and separable, in the theory of factorization and summation respectively (cf. §§6, 7). For factorization into linear primes is ultimate in the sense that the factors cannot be further decomposed into evanescible factors; and linear primes are the only factors with this property. Similarly, for summation into separable primes, *mutatis mutandis.*

§6.—*Factorization into Linear Primes.*

Any evanescible function of n variables can, in general, be factorized into a product of a minimum number of 2^n n-ary linear primes; and the exceptional cases arise when $\bar{S}_r = 0$, $\bar{S}_{r+1} \neq 0$, and then the function can be factorized into a minimum number of $(2^n - r)$ n-ary linear primes. This proposition, which we

will proceed to prove, gives the reason of the term "of deficiency r" which has been applied to $\phi(x, y, \ldots . t)$ when $\overline{S}_r = 0$. For if $\phi(x, y, \ldots . t)$ be evanescible, the minimum number of n-ary linear prime factors has been reduced by r from that of the general case. Also, a very analogous meaning can be found (cf. §8) when the function is not evanescible.

It will be sufficient to write out the proof for a function of two variables: the method is evidently general.

First, to prove that any evanescible function of two variables can be expressed as a product of four factors, each of which is a secondary linear prime.

Assume
$$\phi(x, y) = \prod_{r=1}^{4} p(a_r, x;\ b_r, y).$$

Then, by comparison of coefficients, we have
$$A = \prod_{r=1}^{4} (\bar{a}_r + \bar{b}_r), \quad B = \prod_{r=1}^{4} (\bar{a}_r + b_r), \quad C = \prod_{r=1}^{4} (a_r + \bar{b}_r), \quad D = \prod_{r=1}^{4} (a_r + b_r).$$

Hence,
$$\bar{A} = \sum_{r=1}^{4} a_r b_r, \quad \bar{B} = \sum_{r=1}^{4} a_r \bar{b}_r, \quad \bar{C} = \sum_{r=1}^{4} \bar{a}_r b_r, \quad \bar{D} = \sum_{r=1}^{4} \bar{a}_r \bar{b}_r.$$

These equations for the coefficients are equivalent to the single equation
$$\bar{A} \prod_{r=1}^{4} (\bar{a}_r + \bar{b}_r) + A \sum_{r=1}^{4} a_r b_r + \bar{B} \prod_{r=1}^{4} (\bar{a}_r + b_r) + B \sum_{r=1}^{4} a_r \bar{b}_r$$
$$+ \bar{C} \prod_{r=1}^{4} (a_r + \bar{b}_r) + C \sum_{r=1}^{4} \bar{a}_r b_r + \bar{D} \prod_{r-1}^{4} (a_r + b_r) + D \sum_{r=1}^{4} \bar{a}_r \bar{b}_r = 0. \qquad \text{(m)}$$

Consider this as an equation to find $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$; and put for its left-hand side
$$\lambda(a_1, b_1;\ a_2, b_2;\ a_3, b_3;\ a_4, b_4).$$

Now, since
$$\prod_{r=1}^{4} (\bar{a}_r + \bar{b}_r) = \overline{\sum_{r=1}^{4} a_r b_r},$$

with other analogous equations, we see that in each of the component factors of
$$\prod \lambda \begin{pmatrix} i, i & i, i & i, i & i, i \\ 0, 0 ; & 0, 0 ; & 0, 0 ; & 0, 0 \end{pmatrix},$$

either A or $\bar{A}$, and either B or $\bar{B}$, and either C or $\bar{C}$, and either D or $\bar{D}$, must appear; but not both of any pair.

Again, any one of the four sets of values for a_1 and b_1 given in

$$a_1, b_1 = \left.\begin{matrix} i \\ 0 \end{matrix}\right\},$$

makes one of the four

$$\sum_{r=1}^{4} a_r b_r, \quad \sum_{r=1}^{4} a_r \bar{b}_r, \quad \sum_{r=1}^{4} \bar{a}_r b_r, \quad \sum_{r=1}^{4} \bar{a}_r \bar{b}_r$$

take the value i. Hence, in each of the before-mentioned component factors, at least one out of A, B, C, D must appear as a summand.

Also, it is easy to see that sets of values for a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 can be chosen out of

$$a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 = \left.\begin{matrix} i \\ 0 \end{matrix}\right\},$$

which respectively make three, two, one, none out of $\sum_{r=1}^{4} a_r b_r$, $\sum_{r=1}^{4} a_r \bar{b}_r$, $\sum_{r=1}^{4} \bar{a}_r b_r$, $\sum_{r=1}^{4} \bar{a}_r \bar{b}_r$, vanish.

Hence, the resultant of equation (m) is

$$\begin{aligned}
&(A + B + C + D)(\bar{A} + B + C + D)(A + \bar{B} + C + D)(A + B + \bar{C} + D) \\
&\quad \times (A + B + C + \bar{D})(\bar{A} + \bar{B} + C + D)(\bar{A} + B + \bar{C} + D)(\bar{A} + B + C + \bar{D}) \\
&\quad \times (A + \bar{B} + \bar{C} + D)(A + \bar{B} + C + \bar{D})(A + B + \bar{C} + \bar{D})(\bar{A} + \bar{B} + \bar{C} + D) \\
&\quad \times (\bar{A} + \bar{B} + C + \bar{D})(\bar{A} + B + \bar{C} + \bar{D})(A + \bar{B} + \bar{C} + \bar{D}) = 0. \qquad \text{(n)}
\end{aligned}$$

This reduces to $$ABCD = 0.$$

But this is merely the condition for evanescibility. Hence, equation (m) can always be satisfied when $\phi(x, y)$ is evanescible.

Secondly, to prove that the evanescible function $\phi(x, y)$ can only be expressed as the product of three secondary linear primes when its coefficients satisfy

$$\bar{S_1} + S_4 = 0; \qquad (16)$$

that is, when the function, in addition to being evanescible, is of deficiency one at least.

For, assume $$\phi(x, y) = \prod_{r=1}^{3} p(a_r, x;\ b_r, y).$$

Then, by comparison of coefficients, we have just as in the previous proposition

$$\bar{A}\prod_{r=1}^{3}(\bar{a}_r+\bar{b}_r)+A\sum_{r=1}^{3}a_r b_r+\bar{B}\prod_{r=1}^{3}(\bar{a}_r+b_r)+B\sum_{r=1}^{3}a_r\bar{b}_r$$
$$+\bar{C}\prod_{r=1}^{3}(a_r+\bar{b}_r)+C\sum_{r=1}^{3}\bar{a}_r b_r+\bar{D}\prod_{r=1}^{3}(a_r+b_r)+\bar{D}\sum_{r=1}^{3}\bar{a}_r\bar{b}_r=0. \qquad \text{(p)}$$

The discussion of the form of the resultant of this equation is in all respects similar to that of the form of the resultant of equation (m), except that now it is not possible to choose a set of values out of

$$a_1,\, b_1,\, a_2,\, b_2,\, a_3,\, b_3 = \left.\begin{matrix} i \\ 0 \end{matrix}\right\},$$

which makes none out of $\sum_{r=1}^{3} a_r b_r$, $\sum_{r=1}^{3} a_r \bar{b}_r$, $\sum_{r=1}^{3} \bar{a}_r b_r$, $\sum_{r=1}^{3} \bar{a}_r \bar{b}_r$ to vanish. Thus the resultant of equation (p) is equation (n) with the first factor, namely, $(A+B+C+D)$, omitted. This resultant reduces to equation (16) given above.

Thus we have found the condition that the general minimum number of four secondary linear prime factors may be reduced to three. This condition may be stated: a function of which the field is not restricted,* necessarily has both deficiency and supplemental deficiency.

It is easily proved, by solving for A in equation (16), that a general expression for an evanescible deficient function of two variables is

$$\{\bar{B}\bar{C}\bar{D}+q\,(\bar{B}+\bar{C}+\bar{D})\}\,xy+Bx\bar{y}+C\bar{x}y+D\bar{x}\,\bar{y}.$$

The evanescible function $\phi(x, y)$ can only be expressed as a product of two secondary linear primes when its coefficients satisfy

$$\bar{S}_2+S_4=0\,; \qquad (17)$$

that is, when, in addition to being evanescible, it is of deficiency two at least.

For, assume
$$\phi(x\,,y)=\prod_{r=1}^{2}p\,(a_r,\,x\,;\,b_r,\,y)\,.$$

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* Cf. my "Universal Algebra," §33.

Then, by comparison of coefficients, we have

$$\bar{A}\prod_{r=1}^{2}(\bar{a}_r+\bar{b}_r)+A\sum_{r=1}^{2}a_r b_r+\bar{B}\prod_{r=1}^{2}(\bar{a}_r+b_r)+B\sum_{r=1}^{2}a_r\bar{b}_r$$
$$+\bar{C}\prod_{r=1}^{2}(a_r+\bar{b}_r)+C\sum_{r=1}^{2}\bar{a}_r b_r+\bar{D}\prod_{r=1}^{2}(a_r+b_r)+D\sum_{r=1}^{2}\bar{a}_r\bar{b}_r=0. \quad \text{(q)}$$

The discussion of the form of the resultant of this equation is in all respects similar to that of equations (m) and (p), except that now it is not possible to choose a set of values out of

$$a_1,\ b_1,\ a_2,\ b_2=\left.\begin{matrix} i \\ 0 \end{matrix}\right\},$$

which makes none, or only one, of $\sum_{r=1}^{2}a_r b_r$, $\sum_{r=1}^{2}a_r\bar{b}_r$, $\sum_{r=1}^{2}\bar{a}_r b_r$, $\sum_{r=1}^{2}\bar{a}_r\bar{b}_r$, to vanish. Hence, the resultant of equation (q) is

$$(\bar{A}+\bar{B}+C+D)(\bar{A}+B+\bar{C}+\bar{D})(A+B+C+\bar{D})(A+\bar{B}+\bar{C}+D)$$
$$\times(A+\bar{B}+C+\bar{D})(A+B+\bar{C}+\bar{D})(A+\bar{B}+\bar{C}+\bar{D})(\bar{A}+B+\bar{C}+\bar{D})$$
$$\times(\bar{A}+\bar{B}+C+\bar{D})(\bar{A}+\bar{B}+\bar{C}+D)=0.$$

This reduces to equation (17).

It has already been proved (cf. equation (4)) that the condition that the function $\phi(x, y)$ may be a secondary linear prime is

$$\bar{S}_3+S_4=0.$$

It is easily proved from equation (17) that a general expression for an evanescible function of deficiency, two at least is

$$\{p\bar{q}+(p+\bar{q})(\bar{C}+\bar{D})+\bar{C}\bar{D}\}\,xy+(\bar{p}+\bar{C}\bar{D})\,\bar{x}\bar{y}+C\bar{x}y+D\bar{x}\bar{y}.$$

It must be carefully noticed that, to take the general case, $\phi(x, y)$ can be expressed as the product of four secondary linear primes in an indefinite number of ways: also, that it can be expressed as the product of more than four such primes.

§7.—*Expression of Functions as Sums of Separable Primes.*

This article is devoted to the enunciation of the theorems reciprocal to those of the previous paragraph.

Any function of n variables, capable of the value i, can, in general, be expressed as a sum of a minimum number of 2^n n-ary separable primes; and the exceptional cases arise when $S_{2^n-r+1}=0$ and $S_{2^n-r}\neq 0$, and then the function can be expressed as a sum of a minimum number of (2^n-r) n-ary separable primes.

This proposition gives the reason for the term "of supplemental deficiency r" which has been applied to $\phi(x, y, \ldots. t)$ when $S_{2^n-r+1}=0$. For if $\phi(x, y, \ldots. t)$ be capable of the value i, the general minimum number of n-ary separable prime summands has been reduced by r from that of the general case. The term has also an analogous meaning (cf. §8) when $\phi(x, y, \ldots. t)$ is not capable of the value i.

The special enunciations for functions of two variables are as follows:

Any function capable of the value i can be expressed as a sum of four secondary separable primes.

The conditions that a function can be expressed as a sum of three, or two, secondary separable primes are respectively

$$\bar{S}_1 + S_4 = 0, \tag{18}$$

or

$$\bar{S}_1 + S_3 = 0. \tag{19}$$

The condition that a function may be a secondary separable prime has already (cf. equation (5)) been found to be

$$\bar{S}_1 + S_2 = 0.$$

By a comparison of equations (16) and (18), we deduce the proposition that any function of n variables which can be expressed as a product of (2^n-1) n-ary linear primes, can also be expressed as a sum of (2^n-1) n-ary separable primes, and conversely.

§8.—*Non-Evanescible and Non-Deficient Functions.*

A non-evanescible function can be expressed as the sum of a constant summand and of an evanescible summand. The constant summand is definitely determined, but the general form of the evanescible summand has an ambiguity (that is, an arbitrary element) in its expression.

For, put
$$\phi(x, y) = H + \phi_1(x, y),$$
where H is constant, and the coefficients (A_1, B_1, C_1, D_1) of $\phi_1(x, y)$ satisfy

$$A_1 B_1 C_1 D_1 = 0. \tag{a}$$

Then, by comparison of coefficients,

$$A = A_1 + H, \quad B = B_1 + H, \quad C = C_1 + H, \quad D = D_1 + H. \tag{b}$$

Hence, by multiplication,

$$H + A_1 B_1 C_1 D_1 = ABCD,$$

and thence, from (a),

$$H = ABCD. \tag{c}$$

Also, from (b),

$$\left.\begin{aligned} A_1 &= (\bar{H} + u_1) A = (\bar{S}_4 + u_1) A, \\ B_1 &= \qquad\qquad\quad (\bar{S}_4 + u_2) B, \\ C_1 &= \qquad\qquad\quad (\bar{S}_4 + u_3) C, \\ D_1 &= \qquad\qquad\quad (\bar{S}_4 + u_4) D, \end{aligned}\right\} \tag{d}$$

where we find, by substitution in (a),

$$u_1 u_2 u_3 u_4 S_4 = 0.$$

Thus, finally, the general expression in the required form is

$$\left.\begin{aligned} \phi(x, y) = S_4 + \{ (\bar{S}_4 + u_1) Axy + (\bar{S}_4 + u_2) Bx\bar{y} \\ + (\bar{S}_4 + u_3) C\bar{x}y + (\bar{S}_4 + u_4) D\bar{x}\bar{y} = 0, \\ u_1 u_2 u_3 u_4 S_4 + 0. \qquad\qquad \end{aligned}\right\} \tag{20}$$

Let these two parts of $\phi(x, y)$ be called its constant summand and the general form of its evanescible summand.

It has already been proved in §5 that the evanescible summand of $\phi(x, y)$ cannot have a deficiency of higher order than that of $\phi(x, y)$. It is easily seen that the conditions that the evanescible summand may have the same deficiency as $\phi(x, y)$ are respectively as follows, where U_1, U_2, U_3, U_4 are the symmetric functions (cf. §2) of the four terms u_1, u_2, u_3, u_4:

If $\quad \bar{S}_1 = 0$, then $S_4(U_4 + \bar{U}_1) = 0$.

If $\quad \bar{S}_2 = 0$, then $S_4(U_4 + \bar{U}_2) = 0$.

If $\quad \bar{S}_3 = 0$, then $S_4(U_4 + \bar{U}_3) = 0$.

These conditions can always be satisfied by u_1, u_2, u_3, u_4 in an indefinite number of ways. It may be noted that if $\phi(x, y)$ is of deficiency three, its evanescible summand of the same deficiency is a secondary linear prime.

The corresponding theorem for n variables is as follows: It is always pos-

sible to express a function $\phi(x, y, \ldots. t)$ of n variables in the form

$$\phi(x, y, \ldots. t) = S_{2^n} + \prod_{r=1}^{\mu} p(a_r, x; b_r, y; \ldots. ; k_r, t), \tag{21}$$

where μ has the minimum value of 2^n if $\overline{S_1} \neq 0$, and of $(2^n - r)$, if $\overline{S_r} = 0$ and $\overline{S_{r+1}} \neq 0$. It is from this theorem that the term "deficiency" arises.

Reciprocally, it is always possible to express a function $\phi(x, y, \ldots. t)$ of n variables in the form

$$\phi(x, y, \ldots. t) = S_1 \sum_{r=1}^{\mu} \overline{p}(a_r, x; b_r, y; \ldots. ; k_r, t), \tag{22}$$

where μ has the minimum value of 2^n if $S_{2^n} \neq 0$; and of $(2^n - r)$, if $S_{2^n - r+1} = 0$ and $S_{2^n - r} \neq 0$. This theorem exhibits the meaning of the "supplemental deficiency."

Also, in the case of two variables, let v_1, v_2, v_3, v_4 be any arbitrary terms, and V_1, V_2, V_3, V_4 their symmetric functions. Then the reciprocal theorem to equation (20) is that $\phi(x, y)$ can be expressed in the form

$$\left.\begin{aligned} \phi(x, y) = S_1 \{(\overline{S_1} v_1 + A) xy + (\overline{S_1} v_2 + B) x\overline{y} \\ + (\overline{S_1} v_3 + C) \overline{x} y + (\overline{S_1} v_4 + D) \overline{x}\overline{y} = 0, \\ \overline{S_1}\, \overline{V_1} = 0; \end{aligned}\right\} \tag{23}$$

so that the first factor is constant and the second is capable of the value i. Also the second factor has the same supplemental deficiency as $\phi(x, y)$, if v_1, v_2, v_3, v_4 are chosen to satisfy the following conditions, which are always possible:

If $\quad S_4 = 0$, then $\overline{S_1}\, (\overline{V_1} + V_4) = 0$.
If $\quad S_3 = 0$, then $\overline{S_1}\, (\overline{V_1} + V_3) = 0$.
If $\quad S_2 = 0$, then $\overline{S_1}\, (\overline{V_1} + V_2) = 0$.

By comparison of the first equations of (20) and (23), the following general theorem is deduced: Let u_1, u_2, u_3, u_4 be any set of four arbitraries, and let U_1, U_2, U_3, U_4 be their symmetric functions; and let v_1, v_2, v_3, v_4 be another set of four arbitraries, and let V_1, V_2, V_3, V_4 be their symmetric functions, then the following equation is an identity:

$$\phi(x, y) = S_4 + \overline{S_1}(S_4 + U_1)\, \phi'(x, y); \tag{24}$$

where A', B', C', D' are the coefficients of $\phi'(x, y)$, S_1'', S_2'', S_3'', S_4'' its invariants, and

$$\left.\begin{aligned} A' &= (\overline{S_1} + S_4\overline{U_1})\,v_1 + (\overline{S_4} + u_1)\,A, \\ B' &= (\overline{S_1} + S_4\overline{U_1})\,v_2 + (\overline{S_4} + u_2)\,B, \\ C' &= (\overline{S_1} + S_4\overline{U_1})\,v_3 + (\overline{S_4} + u_3)\,C, \\ D' &= (\overline{S_1} + S_4\overline{U_1})\,v_4 + (\overline{S_4} + u_4)\,D, \end{aligned}\right\} \tag{25}$$

and, after some algebraic reduction,

$$\left.\begin{aligned} S_1'' &= S_1\overline{S_4} + S_1U_1 + V_1, \\ S_2'' &= S_2\overline{S_4} + S_2U_2 + \overline{S_1}V_2 + S_4\overline{U_1}V_2, \\ S_3'' &= S_3\overline{S_4} + S_3U_3 + \overline{S_1}V_3 + S_4\overline{U_1}V_3, \\ S_4'' &= S_4U_4 + \overline{S_4}V_4 + S_4\overline{U_1}V_4. \end{aligned}\right\} \tag{26}$$

Thus it is always possible to choose the arbitraries so that $S_1'' = i$ and $S_4'' = 0$ simultaneously. But it is not possible to make S_2'' have the value i unless $\overline{S_1} + S_2 = i$; that is (cf. §2, equation (1)), unless $S_1 = S_2$; and it is not possible to make S_3'' vanish unless $S_3\overline{S_4} = 0$; that is (cf. loc. cit.), unless $S_3 = S_4$.

For instance, by choosing the arbitraries so that

$$U_1 = U_2 = i, \quad U_3 = U_4 = 0, \quad V_1 = V_2 = i, \quad V_3 = V_4 = 0,$$

we find

$$S_1'' = i, \quad S_2'' = S_2 + \overline{S_1}, \quad S_3'' = S_3\overline{S_4}, \quad S_4'' = 0. \tag{27}$$

§9.—*Linear and Separable Functions.*

A function of n variables $x, y, z, \ldots$, which can be expressed in the form

$$ax + b\bar{x} + cy + d\bar{y} + ez + f\bar{z} + \ldots, \tag{28}$$

will be called "linear," and this form will be called its "linear expression."

A function of n variables $x, y, z, \ldots$, which can be expressed in the form

$$(ax + b\bar{x})(cy + d\bar{y})(ez + f\bar{z}) \ldots, \tag{29}$$

will be called "separable," and this form will be called its "separable expression."

The condition that $\phi(x, y)$ may be linear is found by comparing it with the form (28). Hence,

$$A = a + c, \quad B = a + d, \quad C = b + c, \quad D = b + d.$$

Then, by eliminating a, b, c, d, we find

$$S_1(\bar{A}\bar{D}+\bar{B}\bar{C})=0. \tag{30}$$

It will be observed from the form of this condition that linearity is not an invariant property.

Reciprocally, the condition which the coefficients of $\phi(x, y)$ must satisfy in order that the function may be separable is

$$\bar{S}_4(AD+BC)=0. \tag{31}$$

Thus separability is not an invariant property.

By solving equation (30) for A, it can be proved that a general expression for a linear function is

$$(p+\bar{q})(B+C)\,xy+Bx\bar{y}+C\bar{x}y+D\bar{x}\bar{y}. \tag{32}$$

Also, from equation (31), a general expression for a separable function is

$$(p\bar{q}+BC)\,xy+Bx\bar{y}+C\bar{x}y+D\bar{x}\bar{y}. \tag{33}$$

It is easy to verify that a factor of a linear function is not necessarily itself linear, and that a factor of a separable function is not necessarily itself separable. Reciprocally, a summand of a linear or separable function is not necessarily itself linear or separable.

It can be proved that it is always possible to factorize any function $\phi(x, y)$ into a pair of linear factors; and, reciprocally, that it is always possible to express any function $\phi(x, y)$ as the sum of a pair of separable summands.

Also, the condition that every possible factor of $\phi(x, y)$ may be linear, is

$$\bar{B}\bar{C}+\bar{A}\bar{D}=0. \tag{34}$$

Reciprocally, the condition that every possible factor of $\phi(x, y)$ may be separable, is

$$BC+AD=0. \tag{35}$$

If a linear function is deficient, every possible factor is linear. For the conditions are

$$\bar{S}_1=0, \quad S_1(\bar{A}\bar{D}+\bar{B}\bar{C})=0.$$

Hence,
$$\bar{A}\bar{D}+\bar{B}\bar{C}=0.$$

Reciprocally, if a separable function has supplemental deficiency, every possible summand is separable.

If the function $\phi(x, y)$ is linear, then $S_1 = S_2$. For, by equation (30), we have

$$S_1 = u^{-}(A\bar{D} + \bar{B}\bar{C}) = u(A + D)(B + C).$$

But $$(A + D)(B + C) \mathrel{=\!\!\!\!\!\!\in} S_2.$$

Hence, $$S_1 \mathrel{=\!\!\!\!\!\!\in} S_2.$$

But by §2, $$S_2 \mathrel{=\!\!\!\!\!\!\in} S_1.$$

Hence, $$S_1 = S_2.$$

It follows as a corollary that, if a linear function is deficient, it is of deficiency two at least.

Reciprocally, if the function $\phi(x, y)$ is separable, then $S_3 = S_4$. Also, if a separable function has supplemental deficiency, it has supplemental deficiency two at least.

In general, there are an indefinite number of linear expressions of a linear function. For, let $\phi(x, y)$ be a linear function, and let $ax + b\bar{x} + cy + d\bar{y}$ be one linear expression of $\phi(x, y)$. Then it can be proved that the general form of linear expression of $\phi(x, y)$ is

$$\{(\bar{a} + \bar{b} + u_1)\, a + p\,(ab + cd)\}\, x + \{(\bar{a} + \bar{b} + u_2)\, b + p\,(ab + cd)\}\, \bar{x}$$
$$+ \{(\bar{c}+\bar{d}+v_1)\, c+(\bar{p}+q)(ab+cd)\}\, y + \{(\bar{c}+\bar{d}+v_2)\, d+(\bar{p}+q)(ab+cd)\}\, \bar{y}. \quad (36)$$

But if $\phi(x, y)$ is evanescible, then evidently

$$ab = 0, \quad cd = 0.$$

Hence, each of the coefficients in (36) reduces to the corresponding coefficient of the given linear expression. Thus, when a linear function is evanescible, it has only one linear expression.

[*To be continued.*]

On a Special Form of Annular Surfaces.

By Virgil Snyder.

The equation of a scroll which contains an m-fold linear directrix and another n-fold linear directrix, skew to the first one, is of the form

$$\overset{m}{f}(\lambda, \mu) = 0, \tag{1}$$

wherein

$$\lambda = \frac{a_1x + b_1y + c_1z + d_1w}{a_2x + b_2y + c_2z + d_2w}, \quad \mu = \frac{a_3x + b_3y + c_3z + d_3w}{a_4x + b_4y + c_4z + d_4w};$$

the two directrices being $\lambda = 0$, $\lambda = \infty$; $\mu = 0$, $\mu = \infty$.

The annular surface obtained from this scroll by Lie's equations is of the form

$$\begin{vmatrix} W & 0 & Z & -(X+iY) \\ 0 & W & -(X-iY) & -Z \\ a_1 - \lambda a_2 & b_1 - \lambda b_2 & c_1 - \lambda c_2 & d_1 - \lambda d_2 \\ a_3 - \gamma a_4 & b_3 - \gamma b_4 & c_3 - \gamma c_4 & d_3 - \gamma d_4 \end{vmatrix} = 0, \tag{2}$$

which develops into the form

$$S_1\lambda\mu + S_2\lambda + S_3\mu + S_4 = 0, \tag{3}$$

$S_r = 0$ being the equation of a sphere.

The required surface is the envelope of the sphere (3), whose equation contains the variables λ, μ, connected by the relation (1). It is the tact-invariant of (1), (3) in the λ, μ plane, and contains the variables X, Y, Z, W to the degree $4n(m+n-1)$, $n \leqq m$. Thus the spherical image of the quadric is a quartic (Dupin's cyclide), and that of the cylindroid is an annular surface of order 8, though for special values of the coefficients $a_r b_r$, $c_r d_r$, it may be made smaller.

By a linear transformation of the directrices, the form of the scrolls in (1) is not essentially changed. The number and reality of the pinch-points are not changed, and the asymptotic lines will be projective. The new forms substituted in (2) will give very different forms in spherical space. If the directrices belong to the complex

$$c \equiv (xw' - x'w) - (yz' - y'z) = 0,$$

their spherical images will be points, and the annular surface will be binodal. The locus of centers will be in the radical plane of these two point-spheres. If the directrices are conjugate polars with regard to c, the surface reduces to a curve traced on the surface of a sphere. If one directrix is the line $z = 0$, $w = 0$, the annular surface becomes a cone or developable, according as the other directrix does or does not belong to c. Finally, if both directrices belong to c and cut (z, w), they become isotropic planes, and the surface becomes one of revolution. The important characteristic of this transformation is that asymptotic lines on the scroll (1) become lines of curvature on the corresponding surface in sphere-space; hence the same scroll, projectively considered, will suffice to define the lines of curvature of cones, a species of developables, surfaces of revolution and annular surfaces having two spherical directrices. The locus of centers in every case lies on a quadric of revolution, which may become a plane.

When the scroll has two coincident directrices, the equation has the form

$$\sum_{r=0}^{n} \underset{m+n-2r}{\phi(\lambda)} \, . \, \mu^r = 0, \tag{4}$$

wherein $\phi_s(\lambda)$ is a polynomial in λ of degree s. In this equation λ has the same value as before, but μ is now defined by

$$(a_2x + b_2y + c_2z + d_2w)\,\mu = (a_3x + b_3y + c_3z + d_3w) - \lambda\,(a_4x + b_4y + c_4z + d_4w).$$

The spherical image of this scroll is

$$\begin{vmatrix} W & 0 & Z & -(X+iY) \\ 0 & W & -(X-iY) & -Z \\ a_1 - \lambda a_2 & b_1 - \lambda b_2 & c_1 - \lambda c_2 & d_1 - \lambda d_2 \\ a_3 - \lambda a_4 - \mu a_2 & b_3 - \lambda b_4 - \mu b_2 & c_3 - \lambda c_4 - \mu c_2 & d_3 - \lambda d_4 - \mu d_2 \end{vmatrix} = 0,$$

which develops into

$$\mu = \frac{S_1\lambda^2 + S_2\lambda + S_3}{S_4\lambda + S_5}. \tag{5}$$

The envelope of the sphere (5), when λ, μ are connected by (4), is of degree $4n(m+2n-1)$, $(n \leqq m)$ in X, Y, Z, W.

The spherical image of a scroll defined by an m, $n\,(n \leqq m)$ correspondence in a linear congruence is an annular surface of order $4n(m+n-1)$, when the directrices are skew and of order $4n(m+2n-1)$, when they are coincident. These forms still contain irrelevant factors; by making the directrices satisfy certain conditions, e. g., belong to a given complex, any particular s_i can be eliminated, which reduces the order of the surface. The presence of double generators also reduces the order of the surface, in general, to the form

$$4(mn-\delta),$$

wherein δ is the number of double generators.

Thus, the (2, 2) quartic scroll becomes an annular surface of order 16; it is the transformed surface of the binodal cyclide when the nodes are distinct. The lines of curvature are of order 16 from my article in vol. XXII, p. 96, of the *Journal.* Similarly, the cyclide having three distinct conical points is transformed into a surface of order 12, having lines of curvature of order 8 and one double generator.

Each of these surfaces has two lines of curvature of order half that of the general ones, the curve of contact with the two directrices. Special forms of these are obtained by restricting the pinch-points.

The cyclides which have less than two nodes are not annular surfaces unless the node be biplanar or uniplanar, in which cases they can be readily transformed by the method of equation (5).

An illustration of this method is afforded by Cayley's cubic scroll, which can be written in the form

$$xy^2=(x+y)(xz+yw),$$

or

$$\lambda^2=(1+\lambda)\mu. \tag{6}$$

This is the degraded Kummer surface [(13) 2],* as singular surface of a quadratic complex.

Equation (5) becomes

$$\mu(X^2+Y^2+Z^2)-\lambda XW+(\lambda^2+\lambda)ZW=0,$$

* A. Weiler, Ueber die verschiedenen Gattungen der Complexe zweiten Grades. Mathematische Annalen, vol. 7. See pp. 205-6.

and the envelope of this sphere, subject to the condition (6), is

$$(X^2 + Y^2 + Z^2)^2 + 2W(2Z - X)(X^2 + Y^2 + Z^2) + W^2X^2 = 0,$$

which is a cyclide having a uniplanar point at the origin, and is the type [(13) 1] of Loria.† Three lines of the surface pass through the origin and lie in $X = 0$.

The lines of curvature are defined by the intersection with the surface of a tangent pencil of spheres, whose point of contact is the origin ($W \neq 0$), and the common tangent plane is $Y = 0$. They are of order 4. By the transformation in question, this surface becomes an annular surface of order 12, and whose lines of curvature are defined by the intersection with the surface of a pencil of spheres having for circle of intersection (trajectory circle of a special spherical congruence) a line along which the double directrix touches the surface. The other lines of curvature are of order 8.

The relation between the scroll and the annular surface may be made clearer by giving more of the details of the transformation.

From every point of the double directrix (x, y) issues one real generator of the scroll; there are no pinch-points, but for one point on the directrix the generator which issues from that point coincides with the directrix itself.

The director becomes a sphere S, its points become one generation of minimum lines upon it, and the planes through (x, y) projective with points upon it, go into the second generation of S, projective with the first. Passing through each point P of (x, y) on the scroll and lying in its associated plane π through the director, is a generator g. Through the point on S, determined by the minimum lines m_P and m_π, is a sphere G, containing both minimum lines (touching S). The locus of the points of contact is the trajectory circle of S. Upon this circle are no points at which all the lines of curvature touch each other, but at one of them the generating sphere coincides with S. The generating sphere is real for every point of the trajectory circle.

The locus of centers of the generating spheres (one mantle of the curve of centers) is a curve passing through the center of S and lying on the cone of revolution determined by this point and the trajectory circle. The axis of this cone is the curve of centers of the other system. The only plane line of curvature is the plane of the trajectory circle.

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†G. Loria, Recherche intorno alla geometria della sfera.... Memorie della Acc. Reale di Torino, vol. 36, ser. II, 1884. See p. 100. These types have been more minutely classified by Bôcher in his Reihen entwickelung.... Leipzig, 1894.

There is no distinct cyclide which corresponds to the $\overline{(3,\ 1)}$ scroll. In the equation of the cyclide there are 13 constants, while the equation of the Kummer surface contains 18; hence, several distinct types of general annular surfaces are confounded when the points of space are chosen to represent one fundamental complex.

When the cubic scroll is given in the form

$$y^3 + x(zx + yw) = 0,$$

the corresponding cyclide is of the third order and has a uniplanar point. Its equation is

$$4(X^2 + Y^2 + Z^2)(Z - X) = Z^2.$$

Besides the minimum lines $X \pm iY = 0$, $Z = 0$, the real line $X = 0$, $Z = 0$ lies on the surface and passes through the uniplanar point.

Although this is a contact transformation and transforms spheres into spheres such that lines of curvature are covariant, yet the pencil of tangent spheres which defines the lines of curvature on the cyclide does not transform into the pencil of spheres which defines the lines of curvature of the new surface; in fact, the tangent pencil into which the old tangent pencil is transformed plays no part in the transformed surface.*

On p. 253 of volume XXII of the *Journal*, I gave the equation of a [3, 1] scroll whose asymptotic lines are of order three. The spherical image of this surface is of peculiar type. Let

$$\mu = \frac{1 - 3\lambda^2}{3\lambda - \lambda^3}, \quad \lambda = \frac{x}{y}, \quad \mu = \frac{z}{w};$$

the line $x = 0$, $y = 0$ belongs to c, hence, it will be a point; the line $z = 0$, $w = 0$ goes into the plane at infinity, hence, all the generating spheres of the surface become planes which pass through a fixed point, i. e., the surface is a cone. The equation of this cone is found to be

$$(4x^2 + y^2 + 4z^2)^3 - 27z^2y^4 = 0.$$

Its lines of curvature are cut from the surface by the pencil of spheres concentric with its vertex,

$$x^2 + y^2 + z^2 = k^2w^2.$$

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*The remark made by Casey, Phil. Trans., vol. 161, p. 627, that all the lines of curvature of a trinodal cyclide are circles is incorrect. This is only true of a limiting case of Dupin's cyclide, in which two nodes coincide. It is type [(21)(11)] of Loria. In my classification of Dupin cyclides I called this a "pinch" cyclide.

By eliminating z (e. g.) between these two equations, the projection of the lines of curvature on the x, y plane becomes

$$[kw(8k^2 - 9y^2) \pm 3\sqrt{3}xy^2][kw(8k^2 - 9y^2) - 3\sqrt{3}xy^2] = 0.$$

Both surfaces are symmetric about all the axes, hence each curve is of order 6.

By transforming the cone by the above method, keeping the directrices in the point-complex, a binodal annular surface is obtained whose order is 12, and all of its lines of curvature are still of order 6.

If, instead of x, y and z, w, any two lines which cut the latter and which do not belong to c had been chosen for directrices, the spherical image would be an annular surface of order 12 with two plane directors; hence, all of its lines of curvature are of order 6, and are plane curves. The surface is not a surface of revolution. The two sextics which are cut from the surface by these planes have 36 points of intersection; of these, 18 lie in the circle-points of the cutting plane and six lie on the axis of the planes; the other points are either nodes on the surface or points of tangency of the cutting plane. The latter alternative is excluded because it contains a line of curvature and consequently cuts the surface everywhere at the same angle (Joachimstal's theorem). Hence, the surface has a nodal line of order 12. This surface is thus seen to have a remarkable analogy to Dupin's cyclide. The latter is an annular surface of order 4, and all of its lines of curvature of the second system are plane curves, which factor into conics through the circle-points (circles) and cut each other in points on the axis of the two planes. These curves have only four points in common, two at the circle-points and two on the axis; hence Dupin's cyclide has no other nodal line. This transformation is more general than that by reciprocal radii, which it includes as a particular case. It is not, in general, a point transformation, but, as was shown by Lie, it can be expressed in terms of motion, reciprocation and Bonnet's dilatation, the last transforming a point into a sphere. Analytically, it is expressed as the most general linear transformation of the hexaspherical coordinates which leaves the quadratic form $\Pi = 0$ invariant.*

By means of it a whole class of surfaces can be derived which have the same property as that just cited. Let $f(x, y) = 0$ be the equation of any plane curve.

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*Cf. A. Loewy, "Ueber die Transformationen einer quadratischen Form in sich selbst mit Anwendung auf Linien- und Kugelgeometrie." Nova Acta, Leopoldina, vol. 65, pp. 1-66. It is unfortunate that the word dilatation is used in so different senses. Loewy calls this "eine von Ossian Bonnet angegebene reciproke Umformung."

If it be revolved about any line in the plane, it will generate a surface of revolution whose order is, in general, twice that of the curve, and all of whose lines of curvature are plane curves. If, now, this surface be transformed into line space, it will become a scroll contained in a linear congruence having asymptotic lines of order just half that of the general surface of this type. If the directrices of this congruence be transformed into any lines which cut $z = 0$, $w = 0$ but do not belong to c, the characteristic property of the surface remains unchanged. Now let this new surface be transformed back into spherical space. It will be an annular surface of order higher than that of the surface of revolution, unless the curve $f = 0$ has each circle point of its plane for a multiple point of order half that of f. The new annular surface will not be a surface of revolution, but all of its lines of curvature will be plane, and the curve of each section by these planes will break up into two curves. The nodal line is not the transformed curve of the circles, loci of nodes in f. The planes belong to the same axial pencil, the axis being always a finite line. The planes of the circles of the other system of curvature all touch a cylinder whose generators are perpendicular to this line.

CORNELL UNIVERSITY, *May* 29, 1900.

On the Transitive Substitution Groups whose Order is a Power of a Prime Number.

By G. A. Miller.

In a transitive group G of degree n, the subgroup G_1, which contains all the substitutions of G that do not involve a given letter, is of degree $n-\alpha$ ($\alpha \overline{>} 1$), and G_1 is one of n/α conjugate substitutions of G.* Each of these subgroups is, therefore, transformed into itself by αg_1 substitutions of G, g_1 being the order of G_1. These substitutions constitute a group G_2 of order αg_1. When $\alpha > 1$, G_2 contains a constituent of degree α. Since each of the substitutions of G_2 that is not contained in G_1 contains the α letters which G_1 omits, and since the order of G_2 is αg_1, it follows that the said constituent of G_2 is a regular group of order α. In what follows we shall assume that the order G is a power of a prime p^m. As the order of a subgroup G must be a power of the same prime $\alpha g_1 = p^{k'}$; hence, $\alpha = p^k$. This result may be stated as follows:

Theorem I.—*If the order of a transitive group is a power of a prime p^m, the subgroup formed by all its substitutions which omit a given letter omits p^k ($k \overline{>} 1$) letters of the group.*

Any set of conjugate subgroups or substitutions of G is transformed by all the substitutions of G according to a transitive substitution group of order p^β. Hence, Theorem I includes the theorem that all the substitutions of G which transform one of these subgroups or substitutions into itself, must also transform p^k ($k \overline{>} 1$) of its conjugates into themselves. In other words, the substitutions of G which transform one of a set of conjugate subgroups or substitutions into itself constitute a group in which p^k of these conjugates are invariant.† This includes the theorems. Every non-invariant subgroup or substitution of a group of order

*Cf. Netto, "Theory of Substitutions," 1892, p. 84. Also Cauchy, Comptes Rendus, vol. 21, 1845, p. 669.

† Burnside, "Theory of Groups," 1897, p. 65.

p^m is transformed into itself by p^k $(k \geqq 1)$ of its conjugates. A group of order p^{m-1} that is contained in a group of order p^m is invariant. A group of order p^m cannot be generated by one set of its conjugate subgroups.

Let K represent the group formed by all the substitutions in the holomorph* of G which transform the substitutions of G according to its group of cogredient isomorphisms. The order of K is some power of p, and its subgroup formed by all the substitutions that omit a given letter is the group of cogredient isomorphisms of G. From the facts that each letter of this subgroup corresponds to a substitution of G,† and that this subgroup omits p^k $(k \geqq 1)$ letters of G, it follows that G contains p^k invariant substitutions. Hence, the given theorem includes the important theorem, due to Sylow, that every group of order p^m contains invariant operators besides identity.

When G_1 is transitive, it must be holomorphic to the regular constituent of G_2 mentioned above, since G_2 contains at least one other subgroup which is conjugate with G_1 under G. As all of these conjugates are transitive, there can be only two of them. This is only possible when $p = 2$ and $k = \frac{m-1}{2}$. Hence, the

THEOREM II.—*If the subgroup formed by all the substitutions which omit one letter of a transitive group of order p^m is transitive, the order of the group is 2^{2n+1}, n being any integer.*

When the condition of this theorem is satisfied, G_2 is clearly the direct product of G_1 and its other conjugate. Hence, it follows from a known theorem‡ that the number of transitive substitution groups of order 2^{2n+1}, whose largest subgroups of degree lower than the degree of the group are transitive, is equal to the number of regular groups of degree 2^n.

Since each of these groups may be constructed by writing a regular group of order 2^n in two distinct sets of letters and adding to their direct product a substitution of order two which permutes the corresponding letters of its systems of intransitivity, the number of invariant operators of such a group must be the same as the number of such operators in the mentioned regular group of order 2^n. The largest Abelian subgroup that is contained in such a group is clearly the

* Bulletin of the American Mathematical Society, vol. VI, 1900, p. 396.

† Ibid., vol. V, 1899, p. 245.

‡ Quarterly Journal of Mathematics, vol. 28, 1896, p. 207. American Journal of Mathematics, vol. XXI, 1899, p. 306.

direct product of the Abelian subgroups of these regular groups of order 2^n. Hence, the transitive groups of order p^m in which the subgroup formed by all the substitutions which omit a given letter is transitive, constitute an infinite system of groups of order 2^{2n+1} which are completely determined by the groups of order 2^n.

Suppose that G_1 contains k systems of intransitivity. The number of systems of intransitivity of G_2 is then $\overline{\gtrless} k+1$. We shall first show that $p \overline{\gtrless} k+1$. The largest subgroup of G which transforms G_2 into itself must transform G_1 into p^λ $(\lambda \overline{\gtrless} 1)$ of its conjugates, and hence it must contain $k+1-h\,(p-1)(h \overline{\gtrless} 1)$ systems of intransitivity. This proves that $p \overline{\gtrless} k+1$. When G_1 is transitive, $k=1$ and $p=2$ as was observed above.

When $p=k+1$, the largest subgroup of G that transforms G_2 into itself is transitive. Since this transitive subgroup is of the same degree as G and contains the same subgroup that omits one letter, it must be G itself. In this case G_2 contains p similar regular constituents of order α. These results may be stated as follows:

Theorem III.—*If the subgroup formed by all the substitutions which omit one letter of a transitive group of order p^m contains k systems of intransitivity, then $p \overline{\gtrless} k+1$. When $p=k+1$, the transitive constituents of this subgroup are similar and regular.*

Cauchy proved that the symmetric group of degree n contains subgroups of order p^m, where $m=\varepsilon\left(\frac{n}{p}\right)+\varepsilon\left(\frac{n}{p^2}\right)+\varepsilon\left(\frac{n}{p^3}\right)+\ldots\,;\ \varepsilon\left(\frac{a}{b}\right)$ being the largest integer which does not exceed $\frac{a}{b}$.* We proceed to determine when such a group (G) is transitive and to study some of the properties of these groups. Since the symmetric group of degree n contains all the possible substitutions in n letters, G is of degree n whenever $n \equiv 0 \mod p$. The degree of each of the transitive constituents of G must be a power of p, as it is a divisor of p^m. If G were intransitive when $n=p^\beta$, we would have

$$p^{\beta_1}+p^{\beta_2}+p^{\beta_3}\ldots\ldots=p^\beta,$$

$p^{\beta_1}, p^{\beta_2}, p^{\beta_3}, \ldots.$ being the degrees of the transitive constituents of G. Hence, the number of constituents of lowest degree would be a multiple of p. As these constituents would all be similar, we could combine p of them and thus form a

*Cauchy, Comptes Rendus, vol. 21, 1845, p. 844.

transitive constituent of a larger order in the same letters. This is impossible, as $n!$ is not divisible by p^{m+1}. G must, therefore, be transitive when $n = p^\beta$.

It is clear that all the symmetric groups of degrees $p^\beta + \alpha$, $\alpha < p$ contain the same G, and that the G's of all the symmetric groups of degrees $p^\beta + \alpha'$, $\alpha' < p^{\beta+1} - p^\beta$ are the direct products of this G and the G of the symmetric group of degree α'. That is, when α' is $p^\beta (p > 2)$, the corresponding G is the direct product of the G of the symmetric group of degree p^β and its conjugate written in a distinct set of letters; when $\alpha' = 2p^\beta\ (p > 3)$, the corresponding G is the direct product of three such groups; when $\alpha' = lp^\beta + j\,(p > l+1, j < p^\beta)$ the corresponding G is the direct product of $l + 1$ such groups and the largest group whose order is a power of p that is contained in the symmetric group of degree j. Hence, the

THEOREM IV.—*The largest group G of order p^m that is contained in the symmetric group of degree n is transitive whenever $n = p^{m'} + \alpha$, $\alpha < p$, and only then. When this condition is satisfied, G contains a subgroup of order p^{m-1}, which is the direct product of p conjugate transitive groups of order $p^{\frac{m-1}{p}}$. These transitive groups in turn contain subgroups of order $p^{\frac{m-1}{p}-1}$, which are the direct products of p conjugate transitive groups of order $p^{\frac{m-p-1}{p^2}}$, etc.*

COROLLARY I.—*In a group of order p^m, every subgroup whose order exceeds p^{m-n-1} $(m > p^{n-1} + p^{n-2} + \dots + 1)$ is invariant or contains an invariant subgroup of order $\overline{\overline{>}}\, p$.*

COROLLARY II.—*The largest subgroup of order p^m that is contained in any symmetric group contains just p^γ invariant operators, γ being the number of its transitive constituents.*

This theorem was proved above. To see that it involves Corollary I it is only necessary to observe that a group of order g which contains a subgroup of order g_1, which is not invariant nor contains an invariant subgroup of the entire group, can be represented as a transitive substitution group of degree $g \div g_1$.* Corollary II follows from the fact that each of the transitive constituents of the mentioned subgroups of order $p^{\frac{m-1}{p}}$ contains just p invariant operators.

We proceed to give a method by means of which it is possible to construct a transitive group of order p^m (m being any number greater than 2) which con-

* Dyck, Mathematische Annalen, vol. XXII (1883), p. 102.

tains only p invariant operators. Let H represent a regular group of order p^a which contains only p invariant operators and whose quotient group with respect to these invariant operators contains no operator whose order exceeds p, and let H_1 be the conjugate of H which is formed by all the substitutions (in the same letters as are contained in H) which are commutative with every substitution of H.* Since the quotient group of H_1 with respect to its p invariant operators contains no operator whose order exceeds p, H_1 contains a non-Abelian subgroup of order p_3 which includes its p invariant operators. This subgroup and H generate a group of order p^{a+2} which contains only p invariant operators with respect to which its quotient group contains no operator whose order exceeds p. Since it is well known that groups of the given type exist when $a = 3$ or 4,† it follows that they exist for every value of $a > 2$.

It follows from the preceding paragraph that the only value of m for which a group of order p^m must contain more than p invariant operators is two. It is easily seen that this is also the only value of m for which every subgroup of order p^{m-2} is invariant; for if a group of order p^a contains a non-invariant subgroup of order p^{a-2}, any direct product of which this group is a factor must have the same property. Since groups of order p^3 contain non-invariant subgroups of order p, there must be groups of order p^m (m being any integer > 2) which contain non-invariant subgroups of order p^{m-2}.

The following method may be employed to determine all the groups of order p^m, provided all the groups of order p^{m-1} are known. Suppose that a group (R) of order p^m is represented as a regular group. Any one of its subgroups (H) of order p^{m-1} contains p systems of intransitivity which are permuted according to the group of order p by the remaining substitutions of R. Hence, H may be constructed by writing after each substitution of a regular group of order p^{m-1} the same substitution in $p-1$ distinct sets of letters.‡ All of the other substitutions of R are of the form st, where t merely interchanges the corresponding letters of the p systems of H and s transforms each one of these systems into itself.§

Since t is completely determined by H, it is only necessary to consider how s may be selected. When the substitutions of H are transformed by R

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* Jordan, "Traité des substitutions," 1870, p. 60.

† Hölder, Mathematische Annalen, vol. XLIII, 1893, p. 410.

‡ Quarterly Journal of Mathematics, vol. XXVIII, 1896, p. 236.

§ Ibid.

according to a substitution in its group of cogredient isomorphisms, we may assume that s is commutative with each substitution of H. In this case it may evidently be assumed that s involves only the letters of the first system of intransitivity of H, for, if it were otherwise, we could transform st by a substitution which would transform H into itself and also reduce the number of letters in s. Hence, there cannot be more such groups than the number of sets of substitutions in H which are conjugate under its holomorph. This number may sometimes be reduced by the following considerations:

Let $s_1, s_2, s_3, \ldots, s_p$ represent the constituents of a substitution of H, each constituent involving all the letters of one of the transitive constituents of H. From the equations

$$\begin{aligned}(s_1^{p-1}s_3s_4^2s_5^3 \ldots s_p^{p-2})^{-1}ts_1^{p-1}s_3s_4^2s_5^3 \ldots s_p^{p-2} \\ = (s_1^{p-1}s_3 \ldots s_p^{p-2})^{-1}ts_1^{p-1}s_3 \ldots s_p^{p-2}t^{-1}t = s_1^{1-p}s_2s_3s_4 \ldots s_pt,\end{aligned}$$

it follows that s may be so selected that it is not the p^{th} power of any substitution in the first transitive constituent of H. Since s must clearly be commutative with every substitution of H, none of the powers of a non-commutative substitution in the first transitive constituent of H is a suitable value of s. In particular, we observe that s can have only one value when H is cyclical or when it is Abelian and of type $(1, 1, 1, \ldots)$. It has been assumed throughout that s' is not identity, and that R transforms H according to a substitution in its group of cogredient isomorphisms.

When some substitutions of R transform H according to a substitution which is not in its group of cogredient isomorphisms, we may write these substitutions in the form st_1t, where t_1 and t are commutative, while s is commutative with every substitution of H. As in the preceding case, we may assume that s involves only the letters of the first transitive constituent of H, and hence it is also commutative with t_1. It is evident that t_1 may be restricted to at most one out of each conjugate set of operators of order p^β in the group of isomorphisms of H.

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Geometry on the Cubic Scroll of the Second Kind.

BY FREDERICK C. FERRY.

It is the object of the present paper to give a detailed treatment of several of the more interesting questions of the geometry on the cubic scroll of the second kind, and especially to consider the surfaces which can be passed through any curve on this scroll, so far as the orders of those surfaces and the natures of the residual intersections are concerned. Among published articles, those bearing most directly on the subject proposed are perhaps the following:

Clebsch, "Die Geometrie auf den Flächen dritter Ordnung." Kronecker J., LXV., 359–380.

Chasles, "Ueber die Steiner'sche Fläche." Kronecker J., LXVII., 1–22.

Chasles, "Bemerkung über die Geometrie auf den windschiefen Flächen dritter Ordnung." Math. Ann., I., 634–636.

Voss, "Zur Theorie der windschiefen Flächen." Math. Ann., VIII., 54–135.

Halphen, "Sur la classification des Courbes algébriques." J. de l'Ec. Pol., LII.

Noether, "Zur Grundlegung der Theorie algebraischen Raumcurven." Berlin, 1883.

Sturm, "Ueber die Curven auf der allgemeinen Fläche dritter Ordnung." Klein, Math. Ann., XXI., 457–515.

Rohn, "Die Raumcurven auf den Flächen dritter Ordnung." Leipz. Ber., XLVI., 84–119.

And, in general, the methods employed and the results obtained in the following pages will be found to be analogous to those presented in a paper by the present writer on the "Geometry on the Cubic Scroll of the First Kind," published in the Archiv for Mathematik og Naturvidenskab, B. XXI., Nr. 3.

I.—*The Cubic Scroll of the Second Kind.*

The equation in homogeneous coördinates of the cubic scroll of the second kind is

$$y^3 - x(ys + xz) = 0.$$

Let this equation, or the surface itself, as the case may demand, be denoted by Σ. The double line on Σ is given by $x = 0$, $y = 0$; the plane $x = 0$ touches Σ along the entire length of the double line, and hence contains that line three times; every other plane through the double line contains that line twice, and every plane through the double line cuts out a generator from Σ, the generator cut out by the plane $x = 0$ being coincident with the double line. The plane $y = 0$ is not a determinate plane, for the substitution of $y + \theta x$ for y, with a corresponding change in the coördinates z and s, leaves Σ unchanged in form. The planes $z = 0$ and $s = 0$ change with the plane $y = 0$ and are not determined when the plane $y = 0$ is determined, for the substitution of $z + \lambda x$ and $s + \lambda y$ for z and s respectively leaves Σ unchanged in form. The tangent planes to Σ at any point of the double line are $x = 0$ and the tangent plane to the hyperboloid $ys + xz = 0$ at that point; these two tangent planes coincide at the point $x = 0$, $y = 0$, $s = 0$, hence, this point is a pinch-point on Σ.

Any generator on Σ is given by equations of the form $y = \alpha x$, $z = \alpha^3 x - \alpha s$; the generator given by $y = \alpha_1 x$, $z = \alpha_1^3 x - \alpha_1 s$ meets the double line at its point of intersection with the plane $z + \alpha_1 s = 0$, hence, the points $z = -\alpha s$ on the double line correspond to the planes $y = \alpha x$ through the double line, and in particular the point $s = 0$ on the double line corresponds to the plane $x = 0$ through the double line; i. e., the pinch-point corresponds to the plane which is tangent to Σ all along the double line; thus, to each point of the double line corresponds a generator through that point, while to the point $s = 0$ of the double line corresponds a generator coincident with the double line itself. Therefore, it may be said that two generators on Σ meet the double line at every point, of which one coincides with the double line itself, while at the pinch-point both generators coincide with the double line.

To distinguish the two sheets of Σ in the neighborhood of the double line that will be called the first sheet in which the generators at successive points of the double line are distinct, and the other, in which the generator at every point of the double line coincides with the double line, will be called the second sheet.

II.—*Coördinates on* Σ.

Assuming $x/y=\lambda/\mu$ and $y/z=\lambda/\nu$ and inserting these values in Σ, there results $s/y=\dfrac{\mu^2-\lambda\nu}{\lambda\mu}$, whence the coördinates λ, μ, ν on Σ are connected with the coördinates x, y, z, s on Σ by the relations

$$x:y:z:s=\lambda^2:\lambda\mu:\mu\nu:\mu^2-\lambda\nu.$$

Along the double line in the first sheet, ν/λ and ν/μ take infinite values, while λ/μ is finite, and any generator is given by an equation of the form $a\lambda+b\mu=0$; hence, any homogeneous equation in λ and μ of degree n represents n generators lying in the first sheet in the neighborhood of the double line; and, in particular, $\lambda=0$ gives the generator at the pinch-point, and $\mu=0$ the generator which is cut from Σ by the plane $y=0$. Given, then, a homogeneous equation in λ, μ, ν representing a curve on Σ, to determine the points where this curve or branches of this curve lying in the first sheet in the neighborhood of the double line meet the double line, it is necessary only to put $\nu/\lambda=\nu/\mu=\infty$ or $\lambda/\nu=\mu/\nu=0$ in the given equation, whence there will result a homogeneous equation in λ and μ giving the generators at the points desired.

The double line in the second sheet is given by $\lambda=0$, and to its points correspond finite values of the ratio μ/ν; thus, $a\mu+b\nu=0$ gives a point on the double line in the second sheet; and, given a homogeneous equation in λ, μ, ν representing a curve on Σ, to determine the points where this curve or branches of this curve lying in the second sheet in the neighborhood of the double line meet the double line, it is necessary only to put $\lambda=0$ in the given equation and to solve the resulting equation in μ and ν.

Any plane $ax+by+cz+ds=0$ cuts from Σ a cubic curve whose equation in λ, μ, ν is found by direct substitution to be

$$a\lambda^2+b\lambda\mu+c\mu\nu+d(\mu^2-\lambda\nu)=0.$$

This curve is found by the methods given above to meet the double line in the first sheet where that line is met by the generator $c\mu-d\lambda=0$ and to meet the double line in the second sheet at the point determined by the equation $c\nu+d\mu=0$; now these two points, one lying in the first sheet and the other in the second sheet, are known to be coincident since they are both cut from the double line by the same plane; hence, to the point $d\mu+c\nu=0$ on the double line in the second sheet corresponds by coincidence that point on the double line

in the first sheet which lies on the generator $d\lambda - c\mu = 0$. So, in general, the change of μ to λ and of ν to $-\mu$ in the equation determining a point or points on the double line in the second sheet gives the equation of the generator or generators passing through the same point or points on the double line regarded as lying in the first sheet. Thus the point on the double line in the second sheet given by $\mu = 0$ lies at the pinch-point in the first sheet, and the point on the double line in the second sheet given by $\nu = 0$ lies at the point where the generator $\mu = 0$ meets the double line in the first sheet. In general, a point on the double line will be said to lie in the first sheet or the second sheet according as the equation by which it is determined is of the form $a\lambda + b\mu = 0$ or $a\mu + b\nu = 0$; thus, a homogeneous equation in λ and μ of degree n determines n points on the double line in the first sheet of Σ, and, similarly, a homogeneous equation of degree n in μ and ν determines n points on the double line in the second sheet of Σ.

III.—*Curves on* Σ.

It is evident that, in general, proper curves on Σ are given by irreducible homogeneous equations in λ, μ, ν. Let such an equation, or the curve represented thereby, as the case may demand, be denoted by ϕ; let the degree of ϕ in all three variables be denoted by p and its degree in the variable ν by q, whence it follows that $p \geqq q$. If the terms of ϕ be arranged according to the powers of ν, thus:

$$\phi \equiv U_p + \nu U_{p-1} + \nu^2 U_{p-2} + \ldots\ldots + \nu^\kappa U_{p-\kappa} + \ldots\ldots + \nu^q U_{p-q} = 0,$$

where $U_{p-\kappa}$ denotes a homogeneous polynomial in λ and μ of degree $p-\kappa$, then, as has been shown, $U_{p-q} = 0$ must give at once the $p-q$ generators meeting the double line in the points where that line is met by the curve ϕ in the first sheet; hence, the curve ϕ has $p-q$ points of intersection with the double line in the first sheet of Σ.

To find the points of intersection of the curve ϕ with any generator $a\lambda + b\mu = 0$, it is necessary to substitute $-b/a \,.\, \mu$ for λ in ϕ; having performed this substitution, the equation is divisible by a factor μ^{p-q}; this factor having been rejected, there is left an equation of degree q in μ and ν, which determines q points where the curve ϕ meets the generator in question. Hence, ϕ has q points of intersection with any generator; thus, ϕ has q points of intersection with the double line in the second sheet, which points are found by equating λ

to zero in ϕ and solving the resulting equation in μ and ν, after rejecting the factor μ^{p-q}. If ϕ be arranged according to powers of λ, thus:

$$\phi \equiv V_p + \lambda V_{p-1} + \lambda^2 V_{p-2} + \ldots + \lambda^\theta V_{p-\theta} + \ldots + \lambda^\iota V_{p-\iota} = 0, \quad (\iota \leqq p),$$

where $V_{p-\theta}$ denotes a homogeneous polynomial in μ and ν of degree $p-\theta$, then will $V_p \equiv \mu^{p-q}$. V'_q and $V'_q = 0$ gives the q points where the double line is met by the curve ϕ in the second sheet. Since, then, the curve ϕ meets the double line in $p-q$ points in the first sheet and in q points in the second sheet, it has in all $p-q+q=p$ points of intersection with the double line regarded as lying in both sheets. Any plane through the double line of Σ contains, in addition to that line, a generator of Σ; such a plane meets the curve ϕ in p points on the double line and in q points on the generator in question, and hence contains $p+q$ points of intersection with the curve; hence, the order of the curve ϕ is $p+q$, which will be denoted by m, so that always $m=p+q$. The curve ϕ will often be designated as a (p, q), where p and q have the meanings assigned above. Thus any generator is a $(1, 0)$, the double line regarded as lying in the first sheet is a $(0, 1)$ and regarded as lying in the second sheet is a $(1, 0)$ like any other generator, while, considered as lying in both sheets, it may be said to be a $(1, 1)$, a conic. These results agree entirely with those given in the consideration of the geometry on the cubic scroll of the first kind, if Σ be regarded as obtained from that scroll by allowing the linear director to tend to coincide with the double line in the one sheet, while a generator tends to coincide with the double line in the other sheet, the former coincidence determining the first sheet of Σ and the latter the second sheet of Σ; thus, on the cubic scroll of the first kind, each generator is a $(1, 0)$, the linear director is a $(0, 1)$, and the double director a $(1, 1)$.

Two curves, (p, q) and (p', q'), are said to belong to the same *species* when $p=p'$ and $q=q'$. The number of distinct species of curves of any order m is evidently the greatest integer in $m/2$, the species $(p, 0)$, consisting in each case of only a certain number of generators, being omitted.

A curve ϕ will be said to have a *pair* of branches at any point of the double line when two of its branches lying, one in the one sheet and the other in the other sheet in the neighborhood of the point, intersect at that point of the double line. With reference to the arrangements of the terms of ϕ, according to powers of ν and of λ respectively, as given above, the condition that ϕ have a pair of branches at the point where any generator $a\lambda + b\mu = 0$ meets the double line is

that $V_p \equiv (a\mu - b\nu) . V'_{p-1}$ when $U_{p-q} \equiv (a\lambda + b\mu) . U'_{p-q-1}$, and that ϕ have a pair of branches wherever it meets the double line, it is necessary and sufficient that

$$V_p \equiv C_1 (a_1\mu - b_1\nu)^{\alpha_1} . (a_2\mu - b_2\nu)^{\alpha_2} \ldots . (a_\tau\mu - b_\tau\nu)^{\alpha_\tau} . \mu^{p-q},$$

when $$U_{p-q} \equiv C_2 (a_1\lambda + b_1\mu)^{\beta_1} . (a_2\mu + b_2\mu)^{\beta_2} \ldots . (a_\tau\lambda + b_\tau\mu)^{\beta_\tau},$$

where $\alpha_r \geqq 1$, $\beta_r \geqq 1$, $\alpha_1 + \alpha_2 + \ldots . + \alpha_\tau = q$ and $\beta_1 + \beta_2 + \ldots . \beta_\tau = p - q$. Here can occur several pairs of branches at any point where ϕ meets the double line, and superfluous branches in addition to the number sufficient to make up the pair or pairs of branches at any point can occur. If $p = 2q$, just as many of these superfluous branches will occur in the one sheet as in the other; and if, further, $\alpha_r = \beta_r$ while $p = 2q$, then will only pairs of branches be found without superfluous branches; while in this case, if $\alpha_r = \beta_r = 1$, these pairs will all occur singly. It is geometrically evident that $p = 2q$ when only pairs without superfluous branches occur, or when as many superfluous branches occur in the one sheet as in the other, since in either case just as many points of the curve must lie on the double line in the one sheet as in the other, demanding that $p - q = q$.

If $U_{p-q} \equiv (a\lambda + b\mu)^2 . U'_{p-q-2}$, then will two branches of ϕ, both lying in the first sheet in the neighborhood of the double line, intersect the double line at the same point, but without forming thereby a pair of branches as that term has been defined. Similarly, if

$$U_{p-q} \equiv (a\lambda + b\mu)^\alpha . U'_{p-q-\alpha},$$

then will α branches, all lying in the first sheet in the neighborhood of the double line, intersect that line at the same point. Likewise, if

$$V_p \equiv (c\mu + d\nu)^\beta . V'_{p-\beta},$$

then will β branches of ϕ, all lying in the second sheet in the neighborhood of the double line, intersect that line at the same point.

That every point where ϕ meets the double line be a multiple point of ϕ, resulting from the intersection on that line of branches lying either in the same or in different sheets of Σ in the neighborhoods in question, it is necessary and sufficient that

$$U_{p-q} \overline{V}_p \equiv C . \lambda^{p-q} . (a_1\lambda + b_1\mu)^{\beta_1} . (a_2\lambda + b_2\mu)^{\beta_2} \ldots . (a_\sigma\lambda + b_\sigma\mu)^{\beta_\sigma},$$

where $\beta_r \geqq 2$ and $\overline{V}_p$ is what V_p becomes when μ and ν have been changed to λ and $-\mu$ respectively wherever they occur in it. If $\beta_r = 2$, each point where ϕ meets the double line will be a double point of ϕ, in which case p is evidently even.

If $U_{p-q} \equiv \lambda^a . U'_{p-q-a}$, then will ϕ pass through the pinch-point α times, while each of the α branches lies in the first sheet in the neighborhood of the pinch-point. Similarly, if $V_p \equiv \mu^{p-q+\beta} . V'_{q-\beta}$, then will ϕ pass through the pinch-point β times, while each of the β branches lies in the second sheet in the neighborhood of the pinch-point. And, in general, the curve ϕ has the pinch-point for a point of multiplicity γ when, and only when, $U_{p-q} . \overline{V}_p \equiv \lambda^{p-q+\gamma} . U'_{p-\gamma}$ and the point where any generator $a\lambda + b\mu = 0$ meets the double line is a point of multiplicity γ on the curve ϕ when, and only when,

$$U_{p-q} . \overline{V}_p \equiv \lambda^{p-q} . (a\lambda + b\mu)^{\gamma} . U'_{p-\gamma} .$$

Unless otherwise stated, it will be supposed henceforth that every equation ϕ employed is homogeneous and irreducible.

IV.—*The Curve ϕ as the Intersection, Total or Partial, of Σ with a Surface S.*

To find the equation of a surface S which shall cut the curve ϕ from Σ, it is necessary to substitute x, y, z and s for λ^2, $\lambda\mu$, $\mu\nu$ and $\mu^2 - \lambda\nu$ respectively in the equation ϕ, or in $\omega\phi$, where ω is such a factor as shall render this substitution possible. Evidently, this factor ω must be homogeneous in λ, μ, ν, and let its degree be denoted by n', while m' shall denote the degree of the equation $\omega\phi$ in x, y, z, s after the required substitution has taken place; hence, m' will denote the order of the surface S. If n'_1 denote the degree of ω in the variable ν, then will a residual intersection of order at least as great as $n' + n'_1$ result from the intersection of S and Σ, and, therefore, that the curve ϕ be the complete intersection of S and Σ, it is necessary that $n' + n'_1 = 0$; i. e., it must be true at once that

$$\phi \equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) = 0 ,$$

so that the required substitution may be immediately possible. Since in this case ν enters into ϕ only in one or both of the combinations $\mu\nu$ and $\mu^2 - \lambda\nu$, and since one or both of these combinations must make up a term $C(\mu\nu)^{\alpha} . (\mu^2 - \lambda\nu)^{\beta}$ of ϕ where $\alpha + \beta = 1/2 . p$ (else would ϕ be reducible by the factor λ), it is

clear that in every case of complete intersection $p = 2q$. Again, when

$$\phi \equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) = 0,$$

the intersections of ϕ with the double line in the first sheet are given by an equation which is obtained directly from the above and may be called $f'(\mu\nu - \lambda\nu)/\nu^{\frac{p}{2}} = 0$; similarly, the intersections of ϕ with the double line in the second sheet are given by an equation $f''(\mu\nu, \mu^2)/\mu^{\frac{p}{2}} = 0$; if the divisions by ν and μ be performed, these become respectively $f'(\mu, -\lambda) = 0$ and $f''(\nu, \mu) = 0$; but, since these two equations are made up of the same terms, it is clear that $f' \equiv f''$; it has been shown that two equations $f(\mu, -\lambda) = 0$ and $f(\nu, \mu) = 0$ give the same points of the double line; hence, when ϕ is capable of immediate substitution, not only must $p = 2q$, but also every point where ϕ meets the double line must be a multiple point of ϕ resulting from the occurrence there of a pair or pairs of branches without superfluous branches; the same is geometrically evident whenever the intersection is complete. In general, $m' = 1/2 . (p + n')$, since the required substitution changes a quadratic to a linear factor in every case; hence, when $n' = 0$, it follows that $m' = 1/2 . p$; and, since

$$\tfrac{3}{2} p = p + \tfrac{1}{2} p = p + q = m,$$

no residual intersection can appear, and the curve ϕ is the total intersection of a surface S of order $\frac{1}{2} p$ with Σ when, and only when,

$$\phi \equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) = 0.$$

Thus no curve (p, q), where p is odd, can be the complete intersection of any surface S with Σ. This also is at once evident geometrically.

If $$\phi \not\equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) = 0,$$

it will be necessary to multiply ϕ by such a factor ω of degree n' in λ, μ, ν and n'_1 in ν, where $n' \geqq 1$, as shall render the required substitution possible. The curve ϕ in this case will not be the complete intersection of any surface S with Σ, but there will result a residual intersection of order at least as great as $n' + n'_1$. The order of S being $m' = \frac{1}{2}(p + n')$, m' will be a minimum when n' is a minimum; and those surfaces S of the lowest possible orders, cutting the curves ϕ from Σ, are of special interest, and will be given particular attention. Since, as has been noticed, the required substitution replaces a quadratic by a

linear factor in every case, it is clear that n' must be even when p is even and odd when p is odd.

Since ν can be replaced in $\omega\phi$ only in the combinations $\mu\nu$ and $\lambda\nu$ (the latter occurring in the form $\mu^2 - \lambda\nu$), it is clear that no term of $\omega\phi$ can contain ν to a power greater than the sum of the powers of λ and μ in that term, and this will be true especially of those terms of $\omega\phi$ coming from terms in ϕ which contain the greatest, i. e., the q^{th} power of ν; hence, if $p < 2q$, it will be necessary that ω contain in each of its terms λ or μ, or both, to a power or sum of powers at least as much greater than the degree of ω in ν as is the excess of q over $p - q$; i. e., it must always be true that $p + n' \geqq 2(q + n_1')$; since $m' = \frac{1}{2}(p + n')$, it is now seen that $m' \geqq q + n_1'$, and hence the lowest possible order of S is at least as great as q in every case; it has already been found to be q in the case where (p, q) is the total intersection of S and Σ.

If ω involves ν, then will ω contribute to the residual intersection of S and Σ an irreducible curve of order at least as great as two; while, if ω does not involve ν, it can bring into the residual intersection only generators, n' in number. Therefore, unless otherwise stated, it will be supposed that ω is a homogeneous function involving only λ and μ, and consequently representing n' generators.

Having chosen, then, ω_τ for any given curve ϕ in such a way that $p + \tau \geqq 2q$ and *even*, every term of $\omega_\tau\phi = 0$ can be at once resolved into a constant times a product of powers of $\lambda\nu$, $\mu\nu$, $\lambda\mu$, λ^2 and μ^2; and, further, since

$$\begin{aligned}(\lambda\nu)^\alpha . (\mu^2)^\beta &= (\mu\nu)^{2\beta} . (\lambda\nu)^{\alpha - 2\beta} . (\lambda^2)^\beta && \text{if } \alpha \geqq 2\beta \\ &= (\mu\nu)^\alpha . (\lambda\mu)^{2\beta - \alpha} . (\lambda^2)^{\alpha - \beta} && \text{if } 2\beta \geqq \alpha \geqq \beta \\ &= (\mu\nu)^\alpha . (\lambda\mu)^\alpha . (\mu^2)^{\beta - \alpha} && \text{if } \beta \geqq \alpha,\end{aligned}$$

it follows that this factoring of the terms $\omega_\tau\phi = 0$ can always be done in such a way that no term contains both $\lambda\nu$ and μ^2 as factors. When $\omega_\tau\phi = 0$ has been thus factored in each term, the substitution of x, y, z for λ^2, $\lambda\mu$, $\mu\nu$ respectively, as many times in each term as each of the factors in question occurs there, is at once possible; and thus the equation $\omega_\tau\phi = 0$ is made to involve the variables λ, μ, ν only in the combinations $\lambda\nu$ and μ^2. Since s is to be inserted in place of $\mu^2 - \lambda\nu$ by the required substitution, it is permissible, as a step toward that end, to replace $\lambda\nu$ wherever it occurs in the factored form of $\omega_\tau\phi = 0$ by $\mu^2 - s$, so

that any factor $(\lambda\nu)^\gamma$ now becomes

$$(\lambda\nu)^\gamma = (\mu^2 - s)^\gamma = (\mu^2)^\gamma + C_1(\mu^2)^{\gamma-1}s + \ldots\ldots$$
$$+ C_\kappa(\mu^2)^{\gamma-\kappa}.s^\kappa + \ldots\ldots + C_1\mu^2 s^{\gamma-1} + s^\gamma,$$

containing only s and powers of μ^2. Thus is $\omega_\tau\phi = 0$ made to involve only x, y, z, s and μ^2, and in this form, arranged according to powers of μ^2, it may be written thus:

$$\overline{\omega_\tau\phi} = (\mu^2)^{\frac{1}{2}(p+\tau)}.\psi_0 + (\mu^2)^{\frac{1}{2}(p+\tau)-1}.\psi_1 + \ldots\ldots$$
$$+ (\mu^2)^{\frac{1}{2}(p+\tau)-\kappa}.\psi_\kappa + \ldots\ldots + \mu^2.\psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0,$$

where ψ_κ represents a homogeneous polynomial in x, y, z, s of degree κ.

Let θ_1 and θ_2 denote the number of times that λ and μ respectively occur as factors of all the terms of ω_τ, so that $\omega_\tau \equiv \lambda^{\theta_1}.\mu^{\theta_2}.\omega'_{\tau-\theta_1-\theta_2}$; and let it be assumed for the present that $\theta_1 = 0$. Since the variables x and y can occur in the above equation only from the substitution of the same for the combinations λ^2 and $\lambda\mu$ respectively, it is clear that that equation can contain neither of those variables as a factor unless the equation $\omega_\tau\phi = 0$ contain λ as a factor; that case has now been excluded, for by supposition ϕ is irreducible and $\theta_1 = 0$. Similarly, since the variables z and s enter above only from the substitution of z and $\mu^2 - s$ respectively for $\mu\nu$ and $\lambda\nu$, it follows that the above equation $\overline{\omega_\tau\phi} = 0$ can have neither z nor s as a factor; for ν is not a factor of all the terms of ϕ and does not occur at all in ω_τ. If μ^2 occur ρ times as a factor of the equation in question, it may be removed by dividing by $\mu^{2\rho}$ and making a corresponding reduction in θ_2. But such a limit can be found for θ_2 as shall make the occurrence of μ^2 thus as a factor of this equation impossible, and in the following manner: Among the terms of ϕ is one at least without μ, and consequently of the form $C.\lambda^{p-a}.\nu^a$ where $0 \leqq a \leqq q$; such a term, when multiplied by the term of ω_τ, which is of the lowest degree in μ and may be expressed as $C'.\lambda^{\tau-\theta_2}.\mu^{\theta_2}$, will lead to one of the terms of lowest degree in μ^2 in the derived equation; and, if the factoring and substitution be performed thus:

$$C.\lambda^{p-a}.\nu^a.C'.\lambda^{\tau-\theta_2}.\mu^{\theta_2} \equiv C''.(\mu\nu)^a.(\lambda\mu)^{p+\tau-a-\theta_2}.(\mu^2)^{\theta_2-\frac{1}{2}(p+\tau)} \equiv (\mu^2)^{\theta_2-\frac{1}{2}(p+\tau)}.\psi_{p+\tau-\theta_2},$$

if $\theta_2 \geqq \frac{1}{2}(p+\tau)$; or thus:

$$\equiv C''.(\mu\nu)^a.(\lambda\mu)^{\theta_2-a}.(\lambda^2)^{\frac{1}{2}(p+\tau)-\theta_2} \equiv \psi'_{\frac{1}{2}(p+\tau)},$$

if $a \leqq \theta_2 \leqq \frac{1}{2}(p+\tau)$; or thus:

$$\equiv C''.(\mu\nu)^{\theta_2}.(\lambda\nu)^{a-\theta_2}.(\lambda^2)^{\frac{1}{2}(p+\tau)-a-\theta_2} \equiv \psi''_{\frac{1}{2}(p+\tau)},$$

if $\theta_2 \leqq \alpha$; and only in the first case does the term in question give rise to a term or group of terms containing μ^2 as a factor; therefore, a term which shall not contain the factor μ^2 will always be found in $\overline{\omega_\tau\phi} = 0$ if $\theta_2 \leqq \frac{1}{2}(p+\tau)$; and, in general, if $\theta_2 = \frac{1}{2}(p+\tau)+\gamma$, then will $(\mu^2)^\gamma$ be a factor of every term of the equation $\overline{\omega_\tau\phi} = 0$. Unless otherwise stated, it will be supposed henceforth that $\theta_2 \leqq \frac{1}{2}(p+\tau)+1$.

Among the terms of ϕ is one, at least, without λ and, consequently, of the form $C.\mu^{p-\alpha}.\nu^\alpha$, where $0 \leqq \alpha \leqq q$. Such a term, when multiplied by that term of ω_τ which is of the highest degree in μ and may be written $C'.\mu^\tau$, when the required substitution is performed, gives

$$C.\mu^{p-\alpha}.\nu^\alpha.C'.\mu^\tau = C''.(\mu\nu)^\alpha.(\mu^2)^{\frac{1}{2}(p+\tau)-\alpha} = C''.(\mu^2)^{\frac{1}{2}(p+\tau)-\alpha}.z^\alpha.$$

One or more such terms must always occur in the equation $\omega_\tau\phi = 0$. All the terms in ϕ which are free from λ, together determine, as has been seen, the points where the curve ϕ meets the double line in the second sheet; evidently, the lowest value which α takes in these terms is the same as the order of the multiplicity on the curve ϕ of the point $\lambda = 0$, $\mu = 0$, if that multiplicity have reference here only to the order of intersection at the given point of the branches of ϕ lying in the second sheet in the neighborhood of that point; i. e., if the smallest value which α, as here defined, takes in the equation of the given curve ϕ be denoted by α_1, then will α_1 give the number of times the curve ϕ passes through the point where the generator $\mu = 0$ meets the double line, that point being here regarded as lying in the second sheet. Hence, the equation $\overline{\omega_\tau\phi} = 0$ may always be written thus:

$$\begin{aligned}\overline{\omega_\tau\phi} \equiv (\mu^2)^{\frac{1}{2}(p+\tau)-\alpha_1}.z^{\alpha_1}.\psi_0 + (\mu^2)^{\frac{1}{2}(p+\tau)-1}.\psi_1 + \cdots\cdot \\ + (\mu^2)^{\frac{1}{2}(p+\tau)-\kappa}.\psi_\kappa + \cdots\cdot + \mu^2.\psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0,\end{aligned}$$

where the first term and one, at least, of the last two terms must occur. Since ϕ involves the variable ν in some term or terms to the power q, but no higher power of that variable, and since ω_τ does not involve ν at all, it follows that $\overline{\omega_\tau\phi} = 0$ has likewise in some term or terms the variable ν to the power q, but contains no higher power of that variable. Those terms in $\overline{\omega_\tau\phi} = 0$ involving the q^{th} power of ν have as factors, when that equation has been put in the factored form for substitution, the expression or expressions of the form $(\lambda\nu)^\rho.(\mu\nu)^\sigma$ where $\rho + \sigma = q$; these give rise to terms in $\overline{\omega_\tau\phi} = 0$ involving $s^\rho z^\sigma$, and, con-

sequently, the equation $\overline{\omega_r\phi} = 0$ is of degree q in the variables z, s in some term or terms, but of no higher degree in those variables in any term. Therefore, $\psi_{\frac{1}{2}(p+\tau)-\kappa}$ is of degree at least as great as $\frac{1}{2}(p+\tau)-q-\kappa$ in the variables x, y in every one of its terms.

The equation $\Sigma \equiv y^3 - x(ys+xz) = 0$ may be written $y^3 = x\pi$ if $\pi \equiv ys+xz$. Since the substitution of y^3 for $x\pi$ or of $x\pi$ for y^3 in any or all the terms of the equation of any surface S which cuts the curve ϕ from Σ, is equivalent to replacing S by another surface S' whose equation $S' \equiv S + \psi'_{m'-3}.\Sigma$, where S' contains the entire intersection of S and Σ and is of the same order as S, it is clear that such interchange of y^3 and $x\pi$ in any of the equations now under consideration is allowable; and this means of effecting a change in the equations in question will frequently be employed without further explanation.

Under the conditions of the required substitution, any even power of μ^2 as, e. g., $(\mu^2)^{2\eta}$, can be expressed thus:

$$\begin{aligned}(\mu^2)^{2\eta} &= (\mu^2)^\eta.(s+\lambda\nu)^\eta = (\mu^2 s + \lambda\mu^2\nu)^\eta = (\mu^2 s + yz)^\eta \\ &= (\mu^2)^\eta.s^\eta + \eta.(\mu^2)^{\eta-1}.s^{\eta-1}.yz + \dots + \eta.\mu^2 s.y^{\eta-1}z^{\eta-1} + y^\eta z^\eta,\end{aligned}$$

which, if $\eta \geq 3$, can be expressed otherwise as

$$\begin{aligned}(\mu^2)^{2\eta} = (\mu^2)^\eta s^\eta &+ \eta.(\mu^2)^{\eta-1}s^{\eta-1}.yz + \dots \\ &+ \eta\frac{(\eta-1)}{2}.(\mu^2)^2 s^2.y^{\eta-2}z^{\eta-2} + \frac{\pi}{\lambda^2}xy^{\eta-3}z^{\eta-1}(\eta ys + xz).\end{aligned}$$

Similarly, any odd power of μ^2 as, e. g., $(\mu^2)^{2\eta+1}$, can be expressed thus:

$$(\mu^2)^{2\eta+1} = (\mu^2)^{\eta+1}s^\eta + \eta.(\mu^2)^\eta s^{\eta-1}.yz + \dots + \eta.(\mu^2)^2 s.y^{\eta-1}z^{\eta-1} + \mu^2.y^\eta z^\eta,$$

which, if $\eta \geq 2$, can be put thus:

$$(\mu^2)^{2\eta+1} = (\mu^2)^{\eta+1}s^\eta + \eta.(\mu^2)^\eta s^{\eta-1}.yz + \dots + \eta.(\mu^2)^2 s.y^{\eta-1}z^{\eta-1} + \frac{\pi}{\lambda^2}xy^{\eta-1}z^\eta.$$

By the application of this method of reduction the necessary number of times, any power of μ^2, as, e. g., $(\mu^2)^\theta$. where $\theta \geq 3$, can be brought into the form

$$(\mu^2)^\theta = (\mu^2)^{\gamma_1}.s^{\theta-\gamma_1} + (\mu^2)^4.\psi'_{\theta-4} + (\mu^2)^3.\psi'_{\theta-3} + (\mu^2)^2.\psi'_{\theta-2} + \frac{\pi}{\lambda^2}.\psi'_{\theta-1},$$

where γ_1 has the value 3 or 4 according as $\theta = 2^\kappa + \rho_1$ or $2^\kappa + \rho_2$, while $\rho_1 = 1, 2, 3, \dots 2^{\kappa-1}$ and $\rho_2 = 0, 2^{\kappa-1}+1, 2^{\kappa-1}+2, \dots, 2^\kappa - 1$; $\psi'_{\theta-r}$ denotes a homogeneous polynomial in x, y, z, s of degree $\theta - r$ in all four variables together and of degree at least as great as unity in x, y.

Applying this method of reduction to $\overline{\omega_\tau\phi} = 0$, it is evident that, whenever $\frac{1}{2}(p + \tau) - \alpha_1 \geqq 3$, the term $(\mu^2)^{\frac{1}{2}(p+\tau)-\alpha_1} z^{\alpha_1} . \psi_0$ gives rise to a term or series of terms which, arranged according to descending powers of μ^2, begin thus:

$$(\mu^2)^{\gamma_1} . s^{\frac{1}{2}(p+\tau)-\alpha_1-\gamma_1} . z^{\alpha_1} . \psi_0 + \ldots .$$

and may be written in the form

$$(\mu^2)^{\gamma_1} . (z, s)^{\frac{1}{2}(p+\tau)-\gamma_1} + \ldots .$$

and the entire equation thus finally takes the form

$$\overline{\overline{\omega_\tau\phi}} \equiv (\mu^2)^{\gamma_1} . (z, s)^{\frac{1}{2}(p+\tau)-\gamma_1} + (\mu^2)^4 . \psi''_{\frac{1}{2}(p+\tau)-4} + (\mu^2)^3 \psi''_{\frac{1}{2}(p+\tau)-3}$$

$$+ (\mu^2)^2 . \psi''_{\frac{1}{2}(p+\tau)-2} + \frac{\pi}{\lambda^2} . \psi''_{\frac{1}{2}(p+\tau)-1} + \mu^2 . \psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0$$

The terms making up any one of the groups designated by $\psi''_{\frac{1}{2}(p+\tau)-r}$ may be of two kinds: first, those obtained by the reduction of terms of $\overline{\omega_\tau\phi} = 0$ which involve $(\mu^2)^\theta$, where $\theta \geqq 5$; from the formulæ above it is clear that all terms entering thus in $\psi''_{\frac{1}{2}(p+\tau)-1}$ must be of degree at least as great as two in x, y, and all terms entering thus in $\psi''_{\frac{1}{2}(p+\tau)-2}$ must be of degree at least as great as unity in x, y; and, secondly, in any group $\psi''_{\frac{1}{2}(p+\tau)-r}$, where $r > 1$, there will be included the terms of $\psi_{\frac{1}{2}(p+\tau)-r}$ without change therein; terms in $\psi''_{\frac{1}{2}(p+\tau)-2}$ coming from $\psi_{\frac{1}{2}(p+\tau)-2}$ are known to be of degree as great as $\frac{1}{2}(p + \tau) - q - 2$ in x, y; and this will be as great as unity whenever $\frac{1}{2}(p + \tau) - q \geqq 3$. The first group of terms and one, at least, of the last two groups of terms in $\overline{\overline{\omega_\tau\phi}} = 0$, as written above, will in every case actually occur, and γ_1 can have one or both the values 3 and 4 but no other; in case γ_1 takes both the values 3 and 4, the first group may be broken up, according to the third and fourth powers of μ^2 then occurring, into two groups of terms.

From the equations of substitution, $y^3 = x\pi$, $\lambda^2 = x$, $\lambda\mu = y$, etc., it is easily found that

$$\begin{aligned}
\lambda^2\mu^2 &= y^2, \\
\lambda^2\mu^4 &= \lambda^2 \quad (\mu^2 s + yz) = y^2 s + xyz = y\pi, \\
\lambda^2\mu^6 &= \lambda^2\mu^2 (\mu^2 s + yz) = ys\pi + y^3 z = (ys + xz)\pi = \pi^2, \\
\lambda^2\mu^8 &= \lambda^2 \quad (\mu^2 s + yz) = ys^2\pi + xzs\pi + y^2 z\pi = (s\pi + y^2 z)\pi.
\end{aligned}$$

Consequently, the multiplication of $\overline{\overline{\omega_\tau \phi}} = 0$ by λ^2 will make it possible at once to complete the desired substitution and, therefore, leads to the equation of the surface S in the form

$$S \equiv \pi^2 . (z, s)^{\frac{1}{2}(p+\tau)-3} + \pi . \psi''_{\frac{1}{2}(p+\tau)-1} + y^2 . \psi_{\frac{1}{2}(p+\tau)-1} + x . \psi_{\frac{1}{2}(p+\tau)} = 0 .$$

The order of the surface S is $m' = \frac{1}{2}(p+\tau) + 1$. The first group of terms in the equation as written is of degree two in x, y; the second group is of degree as great as three in x, y if $\frac{1}{2}(p+\tau) - q \geq 3$ and as great as two if $\frac{1}{2}(p+\tau) - q < 3$; the last two groups are of at least the $[\frac{1}{2}(p+\tau) - q + 1]^{\text{th}}$ degree in x, y; and neither the first nor both the last two groups of terms can be wanting in the equation; while, if $\frac{1}{2}(p+\tau) - q = 0$, the occurrence in $\omega\phi = 0$ of a term or terms of the form $(\lambda, \mu)^{p+\tau-q} . \nu^q$ leads to the occurrence in the equation of S of a term or terms of the form $x . (z, s)^{\frac{1}{2}(p+\tau)}$, and, consequently, the last group of terms must occur whenever $\frac{1}{2}(p+\tau) - q = 0$. Hence, if $\frac{1}{2}(p+\tau) - q \geq 1$, the surface S contains the double line of Σ twice and has contact of order $\frac{1}{2}(p+\tau) - q - 1$ with the first sheet of Σ all along that line, while the two sheets of S form a cuspidal edge of contact with the first sheet of Σ all along that line when $\frac{1}{2}(p+\tau) - q \geq 3$. If $\frac{1}{2}(p+\tau) - q = 0$, the surface S contains the double line of Σ once and has contact of the first order with the second sheet of Σ all along that line. In either case, the double line of Σ occurs as a component of the residual intersection $\frac{1}{2}(p+\tau) - q + 1$ times in the first sheet of Σ and twice in the second sheet of Σ, making a total order of $\frac{1}{2}(p+\tau) - q + 3$ for its contribution to the order of the residual intersection. ω_τ introduces τ generators of Σ into the residual intersection, and thus there are already found, as belonging to the residual intersection, straight lines together making up an order of $\frac{1}{2}(p+3\tau) - q + 3$ in that intersection; and, since

$$3\left[\tfrac{1}{2}(p+\tau) + 1\right] - \left[\tfrac{1}{2}(p+3\tau) - q + 3\right] = p + q = m,$$

these lines constitute the entire residual intersection.

The results given above apply to those cases where $\frac{1}{2}(p+\tau) - \alpha_1$ has a value not less than 3; α_1 has been seen to denote the order of the multiplicity of the curve ϕ at that point in the second sheet of Σ where the generator $\mu = 0$ meets the double line, i. e., the number of points of the curve ϕ which lie in the second sheet at that point. This is not a singular point of the surface Σ, and, by a change of coördinates, it is always possible to give such a form to the equation

of the curve ϕ that α_1 shall take a lower value, which lower value may, in general, be made to be zero. For this purpose, it is only necessary to choose for the plane $y=0$ a plane containing no point of the given curve on the double line in the second sheet. In this way all cases where $\frac{1}{2}(p+\tau)-\alpha_1 \leqq 2$ can be referred to those already considered. But for the sake of obtaining some conditions regarding the nature of the surface S, the cases where $\alpha_1 \leqq \frac{1}{2}(p+\tau)-2$ will now be somewhat further investigated.

Given, then, $\alpha_1 \geqq \frac{1}{2}(p+\tau)-2$, it is possible, by the methods employed in the general case, to reduce $\overline{\omega_\tau\phi}=0$ to the form

$$\overline{\overline{\omega_\tau\phi}} \equiv (\mu^2)^{\frac{1}{2}(p+\tau)-\alpha_1}\,.\,z^{\alpha_1}\,.\,\psi_0 + (\mu^2)^4\,.\,\psi''_{\frac{1}{2}(p+\tau)-4} + (\mu^2)^3\,.\,\psi''_{\frac{1}{2}(p+\tau)-3}$$
$$+ (\mu^2)^2\,.\,\psi''_{\frac{1}{2}(p+\tau)-2} + \frac{\pi}{\lambda}\,.\,\psi''_{\frac{1}{2}(p+\tau)-1} + \mu^2\,.\,\psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0,$$

where $\psi_0 \neq 0$, and hence may be removed by division. There now arise three cases, in each of which the equation of the surface S is readily obtained on the multiplication of the above equation by λ^2 and completing the substitution throughout. These three cases are

1. If $\alpha_1 = \frac{1}{2}(p+\tau)-2$,
$$S \equiv y\pi\,.\,z^{\frac{1}{2}(p+\tau)-2} + \pi\,.\,\psi'''_{\frac{1}{2}(p+\tau)-1} + y^2\,.\,\psi_{\frac{1}{2}(p+\tau)-1} + x\,.\,\psi_{\frac{1}{2}(p+\tau)} = 0;$$

2. If $\alpha_1 = \frac{1}{2}(p+\tau)-1$,
$$S \equiv y^2\,.\,z^{\frac{1}{2}(p+\tau)-1} + \pi\,.\,\psi'''_{\frac{1}{2}(p+\tau)-1} + y^2\,.\,\psi_{\frac{1}{2}(p+\tau)-1} + x\,.\,\psi_{\frac{1}{2}(p+\tau)} = 0;$$

in both these cases the first term and one at least of the last two groups of terms will occur, and the surface S contains the double line of Σ twice if $\frac{1}{2}(p+\tau)-q \geqq 1$.

3. If $\alpha_1 = \frac{1}{2}(p+\tau)$,
$$S \equiv x\left[z^{\frac{1}{2}(p+\tau)} + \psi_{\frac{1}{2}(p+\tau)}\right] + \pi\,.\,\psi'''_{\frac{1}{2}(p+\tau)-1} + y^2\,.\,\psi_{\frac{1}{2}(p+\tau)-1} = 0,$$

where the first group of terms is never wanting, and the surface S contains the double line of Σ once. In all three cases $\psi'''_{\frac{1}{2}(p+\tau)-1}$ is of degree as great as unity in x, y; hence, if $\frac{1}{2}(p+\tau)-q \geqq 1$, the surface S contains the double line of Σ twice and has contact of order $\frac{1}{2}(p+\tau)-q-1$ with the first sheet of Σ all along that line; and, if $\frac{1}{2}(p+\tau)-q=0$, the surface S contains the double line of Σ once and has contact of the first order with the second sheet of Σ all along that line. Therefore, here, just as in the general case, the residual intersection is entirely made up of the double line of Σ occurring $\frac{1}{2}(p+\tau)-q+1$

times in the first sheet and twice in the second sheet of Σ, and of generators of Σ to the total order τ introduced by ω_τ.

Since the surface S contains the curve ϕ but once in general, the equation of that surface will not contain Σ, nor the equation of any other second surface through the given curve, as a factor of all its terms; and, since ϕ is by supposition a proper curve and the entire intersection of S with Σ has been found to consist of the curve ϕ and straight lines, it appears that the only cases in which S is an improper surface occur when S is made up of a surface containing the given curve and another surface or surfaces cutting from Σ only straight lines, the double line and generators of Σ. Such supplementary surfaces are made up of planes through the double line of Σ, and, consequently, are represented by equations of the form $ax + by = 0$. Thus S will be, in general, an improper surface only when its equation contains one or more polynomials of the form $ax + by$ as a factor or factors of all its terms. Since these polynomials are each homogeneous in the variables x, y, if the terms of the equation of S be arranged in groups according to the degrees of those terms in x, y, every such group must by itself contain each such polynomial $ax + by$ as a factor; and, since the equation of the surface S is in no case of greater degree than two in x, y, not more than two factors of the required form can ever occur.

If, then, $ax + by$ be a factor of all the terms of the equation of S, the plane $ax + by = 0$ cuts from Σ a generator introduced into the residual intersection by ω_τ, unless $b = 0$; and if any other of the single infinity of generators of Σ be substituted in ω_τ for the generator cut out by the plane in question, the existence of a corresponding factor will, in general, be impossible in the equation of S, and the surface will become in that case a proper surface. To determine in what cases b can have the value zero, let it be supposed that x is a factor of every term in the equation of S; the plane $x = 0$ contains the double line in the second sheet of Σ twice; the residual intersection of S with Σ contains that line in that sheet also twice; if, now, the factor x be removed from the equation of S, leaving the equation of a surface S' of order $\frac{1}{2}(p + \tau)$, the residual intersection of S' with Σ cannot contain the double line in the second sheet of Σ at all; but, if $\frac{1}{2}(p + \tau) - q \geqq 1$, the equation of S being of the second degree in x, y, the equation of S' must be of the first degree in those variables, and hence the double line of Σ must still occur at least once in the residual intersection of S' with Σ; and, if $\frac{1}{2}(p + \tau) - q = 0$, the equation of S' will be of zero

degree in x, y, and, consequently, the residual intersection cannot contain the double line of Σ; on the double line in the first sheet of Σ are $p-q$ points of the given curve and τ points where the τ generators in ω_τ meet that line, while on that line in the second sheet of Σ are q points of the curve only; since the curve and the τ generators must make up the entire intersection of S' with Σ, it is evident that either $\tau = 0$ or else the τ generators are so chosen and the nature of the curve is such that the combined locus of the curve and the τ generators have a pair or pairs of branches wherever it meets the double line; since this latter condition is not, in general, fulfilled (and may always be avoided by appropriate choice of ω_τ if $\tau \neq 0$), it follows that τ must here have the value zero. But in that case $p = 2q$, $m' = \frac{1}{2}(p + \tau) = \frac{1}{2}p$, and the case is that of complete intersection which has been excluded from the cases now under consideration. From another point of view, since the equation of the curve contains a term $\mu^{p-\alpha_1}.\nu^{\alpha_1}$, there must occur in $\omega_\tau\phi = 0$ a term of the form $\mu^{p+\tau-\alpha_1}.\nu^{\alpha_1}$ and x can be a factor of all the terms of the equation of S only if here $p + \tau - \alpha_1 \leqq \alpha_1$; but this condition can be satisfied only when $p + \tau = 2\alpha_1 = 2q$ (since $\alpha_1 = q$), and hence, as explained on page 189, the curve ϕ must have all its branches in the second sheet in the neighborhood of the double line meet that line at the point where the generator $\mu = 0$ meets it; in other words, the point in the second sheet where the generator $\mu = 0$ meets the double line is a q-tuple point of the curve ϕ in that sheet; and not only must every branch of the curve ϕ in the first sheet of Σ in the neighborhood of the double line pass through this same point, making it $(p-q)$-tuple in the first sheet, and hence a p-tuple point on the curve ϕ as a whole, but also must ω_τ have been made up entirely of the generator $\mu = 0$ occurring τ times, which, in general, will not be the case. Hence, the cases of complete intersection being excepted, the equation of the surface S will contain the factor x only when ω_τ has been chosen in a very special form, i. e., when $\omega_\tau = \mu^\tau$ or $C.\mu^\tau$.

It has been seen that, when $\frac{1}{2}(p+\tau) - \alpha_1 \geq 3$, the equation of S has the form

$$S \equiv \pi^2.(z,\, s)^{\frac{1}{2}(p+\tau)-3} + \pi.\psi'''_{\frac{1}{2}(p+\tau)-1} + y^2.\psi_{\frac{1}{2}(p+\tau)-1} + x.\psi_{\frac{1}{2}(p+\tau)} = 0.$$

If $\frac{1}{2}(p+\tau) - q \geq 3$, the only terms occurring here in the second degree in x, y are contained in the first group and involve those variables in the form $(xs + yz)^2$; hence, no factor of the form $ax + by$ can occur in this case and S is a proper surface. If $\frac{1}{2}(p+\tau) - q = 2$, terms of the second degree in x, y occur only in

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the first two groups of terms and all have the factor π; therefore, at most, a single factor $ax + by$ can here be found; if such a factor does occur, the plane $ax + by = 0$ cuts from Σ a generator contained in ω_τ, and the substitution of any other generator in that expression in place of the one there occurring will, in general, lead to a proper surface S. If $\frac{1}{2}(p + \tau) - q = 1$, terms of the second degree in x, y can occur in every group above, and if one or, at most, two factors of the given form occur, other generators may be selected for replacing in ω_τ the one or two cut out by the plane or planes in question and a proper surface S will, in general, be thus obtained. If, finally, $\frac{1}{2}(p + \tau) - q = 0$, the terms of the first degree in x, y all occur in the last group and have x as the only possible factor of the required form; but it has been already seen that a proper surface S can in this case always be found.

There still remain the cases where $\frac{1}{2}(p + \tau) - \alpha_1 \leqq 2$, but these can be referred, by a change of coördinates, to those already considered. It is clear, however, from the forms of the equations of S, as given on page 193, that the results obtained above find immediate application here; and, since $\alpha_1 \leqq q$, only those cases where $\frac{1}{2}(p + \tau) - q \leqq 2$ demand consideration. The equations of S, as well as the fact that the point cut from the double line in the second sheet of Σ by the plane $y = 0$ is a point of multiplicity of order $\alpha_1 \geqq q - 2$ on the curve ϕ in that sheet, suggest that in the cases in question the methods followed, without change of coördinates, may lead to the occurrence of the plane $y = 0$ as a component of the surface S. An upper limit for θ_2, which shall render this impossible, is readily obtained thus: The term of the equation of ϕ, in which λ is wanting and ν appears to the power α_1, when multiplied by that term of ω_τ, which is of the lowest degree in μ, will take the following form for substitution:

$$C_1\mu^{p-\alpha_1}\nu^{\alpha_1}.\, C_2\lambda^{\tau-\theta_2}\mu^{\theta_2} = C'(\mu\nu)^{p-\alpha_1+\theta_2}.\,(\lambda\nu)^{2\alpha_1-p-\theta_2}.\,\lambda^{\tau+p-2\alpha_1},$$

if $\theta_2 \leqq 2\alpha_1 - p$; and this condition for θ_2 is, then, a sufficient one to insure that the equation of the surface S be not reducible by the factor y. This condition, in the three most unfavorable cases, takes the forms

1. $\theta_2 \leqq \tau$, when $\frac{1}{2}(p + \tau) = q$.
2. $\theta_2 \leqq \tau - 2$, when $\frac{1}{2}(p + \tau) = q + 1$.
3. $\theta_3 \leqq \tau - 4$, when $\frac{1}{2}(p + \tau) = q + 2$.

Since the generator $\mu = 0$ is in no way singular on Σ, whatever is true for it may, by a change of coördinates, be made to apply, in general, to any other generator in the first sheet of Σ. And it has been seen that S is a proper surface at once, save only in those cases where ω_τ contains a certain generator or generators, having a particular relation to the curve in question, more than a certain easily ascertainable number of times. From the single infinity of generators on Σ it will always be possible to make up ω_τ in such a way that these exceptional cases may, from the outset, be avoided.

It is now clear that in every case, in general, where τ is so taken that $p + \tau \geqq 2q$, and even, while in addition $\theta_1 = 0$ and $\theta_2 \leqq \frac{1}{2}(p + \tau)$, then can the method given be made to determine the equation of a proper surface S of any desired order $m' = \frac{1}{2}(p + \tau) + 1 \geqq \frac{1}{2}p + 1$. Since a surface S of order $\frac{1}{2}(p + \tau) + 1$ cannot contain a line of greater multiplicity than $\frac{1}{2}(p + \tau)$ without breaking up into surfaces of lower orders, it is evident that S must have contact with Σ of order at least as great as one all along the generator $\mu = 0$ whenever $\theta_2 = \frac{1}{2}(p + \tau) + 1$. Since the generators together make up a total order τ in the residual intersection, there will be contact between S and Σ all along one or more of the generators of Σ in question whenever less than τ different generators enter in ω_τ, unless the occurrence of one or more of the generators in question as multiple lines of S accounts for the entire number of lines of intersection by which τ exceeds the number of different generators involved in ω_τ. In general, then, subject to the conditions already given, ω_τ may be chosen to contain any τ generators whatever lying in the first sheet of Σ, or any lesser number of generators in that sheet to become, in part at least, multiple lines of S, or lines of contact between the two surfaces S and Σ; but, in every case, the total order in the intersection must be τ, no generator can be of multiplicity of order greater than $\frac{1}{2}(p + \tau)$ on S, and any generator occurring more than $\frac{1}{2}(p + \tau)$ times in the intersection is a line of contact of S with Σ.

If it be desired so to choose ω_τ that a reduction by one or two degrees in the order of the surface S be obtained, it is evident that τ must be so chosen as to make $\frac{1}{2}(p + \tau) - q = 0$ or 2 for a reduction by unity, or $\frac{1}{2}(p + \tau) - q = 1$ for a reduction by unity or two. And, further, it must be possible to choose ω_τ in such a way that at every point where a branch of the curve ϕ meets the double line, shall occur a pair or pairs of branches of the locus $\omega_\tau\phi = 0$, if the reduction is to be by two degrees, or if $\frac{1}{2}(p + \tau) - q = 0$ and the reduction is

to be by unity. Sufficient conditions for the reduction of the order of the surface S in the special cases, where alone such reduction is possible, will not be considered further here.

The cases where λ occurs as a factor of all the terms of ω_τ, and where, consequently, $\theta_1 > 0$, can all be solved in a manner entirely similar to that already given for the cases where $\theta_1 = 0$. The term of highest degree in μ arising in $\omega_\tau\phi = 0$ from any term $C.\lambda^{p-\alpha-\beta}.\mu^\alpha.\nu^\beta$ of $\phi = 0$ is of the form $C'.\lambda^{p-\alpha-\beta+\theta_1}.\mu^{\alpha+\tau-\theta_1}.\nu^\beta$, which, when factored in the manner prescribed for substitution, becomes

$$C.(\mu\nu)^\beta.(\lambda\mu)^{p-\alpha-\beta+\theta_1}.(\mu^2)^{\frac{1}{2}(\tau-p)+\alpha-\theta_1}, \text{ if } \theta_1 \leqq \tfrac{1}{2}(\tau-p)+\alpha.$$

Since α must take the value $\alpha \geqq p - q$ in some term of $\phi = 0$, a sufficient condition that some term of $\overline{\omega_\tau\phi} = 0$ in this case contain $(\mu^2)^\varepsilon$, where $\varepsilon \geqq 2$, is that $\theta_1 \leqq \frac{1}{2}(p+\tau) - q - 1$; and, if $\theta_1 = \frac{1}{2}(\tau - p) + \alpha_2 - 1$, where α_2 is the largest value assumed by α in the equation of ϕ, then will μ^2 occur in $\overline{\omega_\tau\phi} = 0$ while μ^4 will not occur in that equation. Hence, the condition $\theta_1 \geqq \frac{1}{2}(p+\tau)$ causes $\overline{\omega_\tau\phi} = 0$ to be at once the equation of the surface S, which may be a proper surface if $\alpha_2 = p$, but will contain as a component the plane $x = 0$ if $\alpha_2 \leqq p - 1$. It will now be supposed that θ_1 is allowed to have any value consistent with the newly obtained condition, viz., that

$$\theta_1 \leqq \tfrac{1}{2}(p+\tau) - q - 1.$$

This condition involves that $p + \tau - \theta_1 \geqq 2q$, and hence the equation $\overline{\omega_{\tau-\theta_1}\phi} = 0$ can, when $p + \tau - \theta_1$ is even, like the equation $\overline{\omega_\tau\phi} = 0$ on page 188, be written in the partially substituted form thus:

$$\begin{aligned}\overline{\omega_{\tau-\theta_1}\phi} \equiv (\mu^2)^{\frac{1}{2}(p+\tau-\theta_1)}.\psi_0 + (\mu^2)^{\frac{1}{2}(p+\tau-\theta_1)-1}.\psi_1 + \dots\\ + (\mu^2)^{\frac{1}{2}(p+\tau-\theta_1)-\kappa}.\psi_\kappa + \dots + \mu^2.\psi_{\frac{1}{2}(p+\tau-\theta_1)-1} + \psi_{\frac{1}{2}(p+\tau-\theta_1)} = 0.\end{aligned}$$

If $p + \tau - \theta_1$ be odd, the equation $\overline{\omega_{\tau-\theta_1}\phi} = 0$ will take the same form, with the exception that an odd instead of an even power of μ is a factor of every group of terms save the last, and that the last group of terms has the form $\lambda.\overline{\psi}_{\frac{1}{2}(p+\tau-\theta_1-1)}$ $+\mu.\overline{\overline{\psi}}_{\frac{1}{2}(p+\tau-\theta_1-1)}$; but in either case multiplication by λ^{θ_1+2}, after the method used when $\theta_1 = 0$, leads to the equation of S in the form

$$S \equiv [\pi^2.(z, s)^{\frac{1}{2}(p+\tau)-\theta_1-3} + \pi.\psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}].y^{\theta_1} + \overline{\psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 \geqq 3$; but in the form

$$S \equiv [y\pi \,.\, z^{\frac{1}{2}(p+\tau)-\theta_1-2} + \pi \,.\, \psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}] \cdot y^{\theta_1} + \overline{\Psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 = 2$; and in the form

$$S \equiv [y^2 \,.\, z^{\frac{1}{2}(p+\tau)-\theta_1-1} + \pi \,.\, \psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}] \cdot y^{\theta_1} + \overline{\Psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 = 1$; and, finally, in the form

$$S \equiv [x \,.\, z^{\frac{1}{2}(p+\tau)-\theta_1} + \pi \,.\, \psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}] \cdot y^{\theta_1} + \overline{\Psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 = 0$.
Just as in the case where $\theta_1 = 0$, so here it can be shown that neither the first nor the last group of terms can fail to occur in the equation of the surface S; and, when $\frac{1}{2}(p+\tau)-q \geqq 1$, the surface S contains the double line of Σ θ_1+2 times, and has contact of order $\frac{1}{2}(p+\tau)-q-\theta_1-1$ with the first sheet of Σ all along that line; similarly, if $\frac{1}{2}(p+\tau)-q=0$, the surface S contains the double line of Σ θ_1+1 times and has contact of the first order with the second sheet of Σ all along that line. Consequently, the residual intersection is here entirely made up in every case, in general, of the double line of Σ occurring $\frac{1}{2}(p+\tau)-q+1$ times in the first sheet and θ_1+2 times in the second sheet of Σ, and of generators of Σ to the total order $\tau-\theta_1$ introduced by $\omega_{\tau-\theta_1}$. Hence, the generator of Σ lying coincident with the double line in the second sheet enters like any other generator into the residual intersection, so long, at least, as θ_1 has any value within the limits assigned; and thus this case can be included in the general one and the results of the considerations of the nature of the surface S, and of the resultant residual intersection in particular, may be extended so as to include all cases, without making any distinction between the generator in the second sheet of Σ and any other generator of that surface, provided only that θ_1 and θ_2 do not exceed the limits $\frac{1}{2}(p+\tau)-q-1$ and $\frac{1}{2}(p+\tau)+1$ respectively. Therefore, there may be stated the following

THEOREM: *A proper surface S, having for its partial intersection with Σ any given proper curve of order m on Σ, can be found of any desired order $m' = \frac{1}{2}(p+\tau)+1$, where $\tau \geqq 0$, and $p+\tau \geqq 2q$ and even, whose residual intersection with Σ shall consist entirely of the double line of Σ occurring $\frac{1}{2}(p+\tau)-q+1$ times in the first sheet of Σ and twice in the second sheet of Σ, together with arbitrary generators of Σ occurring in such manner as to give a total order of τ in the intersec-*

tion, while no particular generator occurs more than $\frac{1}{2}(p+\tau)-q-1$ *times among such arbitrary generators.*

In general, ω_τ must have $\theta_1 \leqq \frac{1}{2}(p+\tau)-q-1$, and, for values of θ_1 which are greater than zero, the generator in the second sheet of Σ must be regarded as occurring θ_1 times in the residual intersection, in addition to the two times given in the theorem. That this line always occurs twice in the residual intersection is the natural result of the multiplication of $\overline{\overline{\omega_\tau \phi}} = 0$ in every case by λ^2. Also must ω_τ have $\theta_2 \leqq \frac{1}{2}(p+\tau)-1$, and, if ω_τ introduce only generators in the first sheet of Σ which do not meet the given curve on the double line of Σ, the particular method of substitution given will lead at once to a proper surface S. Clearly, by varying the form of ω_τ, an infinite number of such surfaces S of any order $m' = \frac{1}{2}(p+\tau)+1$ can be found for any given curve ϕ.

WILLIAMS COLLEGE, *February* 1, 1900.

(*To be continued.*)

If f denote the total order of intersection of the sheet or sheets of S with the two sheets of Σ all along the double line of the latter surface, f_1 denote the component of that total order arising in the first sheet and f_2 likewise the component of that total order found in the second sheet, f'_s ($s = 1$ or 2) denote that component of f_s due to simple or multiple intersection alone (with regard to the sheet indicated by s, or both sheets, if s is not written), regardless of contact of sheets, and f''_s likewise the component due to such contact between sheets of S and Σ, while g_1 and g_2 in like manner denote the total orders, in the residual intersection, of the generators brought in by ω_r in the first and the second sheets of Σ respectively, then will

$$f = f_1 + f_2 = f'_1 + f'_2 + f''_1 + f''_2 = f' + f'' \text{ and } f'_1 = f'_2,$$

while, in general,

$$f = \tfrac{1}{2}(p + \tau) - q + 3, \quad f_1 = \tfrac{1}{2}(p + \tau) - q + 1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1,$$

and $g_2 = \theta_1$ (which last, g_2, is included in f_2 and will henceforth be omitted save as it appears there). The general case may, then, be written thus:

$$m' = \tfrac{1}{2}(p + \tau) + 1, \quad f_1 = \tfrac{1}{2}(p + \tau) - q + 1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1,$$

where $\omega = \lambda^2 . \omega_r$, $p + \tau \geqq 2q$ and *even*, and $\theta_1 \leqq \frac{1}{2}(p + \tau) - q - 1$. And here will

$$f'_1 = f'_2 = 1, \qquad f''_1 = 0, \qquad f''_2 = 1,$$

if $p + \tau = 2q$; while

$$f'_1 = f'_2 = \theta_1 + 2, \quad f''_1 = \tfrac{1}{2}(p + \tau) - q - \theta_1 - 1, \quad f''_2 = 0,$$

if $p + \tau > 2q$.

The cases where S is the surface of lowest order which can be passed through the given curve on Σ, can now be solved readily.

If $\phi \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, the case is that of complete intersection already considered, pages 185 and 186, and may be expressed thus:

I′. $\phi \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $p = 2q$, $m' = \frac{1}{2}p$, $f = g_1 = 0$, $\omega \equiv 1$.

If $\phi \not\equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$ and $p \geqq 2q$ and *even*, the results above show that τ can be given the value zero, leading to the general case

I. $\phi \not\equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $p \geqq 2q$, p *even*, $m' = \frac{1}{2}p + 1$,
$f_1 = \frac{1}{2}p - q + 1$, $f_2 = 2$, $g_1 = 0$, $\omega \equiv \lambda^2$,

while

$$f_1' = f_2' = 1, \quad f_1'' = 0, \quad f_2'' = 1, \text{ if } p = 2q,$$

but $$f_1' = f_2' = 2, \quad f_1'' = \tfrac{1}{2}p - q - 1, \quad f_2'' = 0, \text{ if } p > 2q.$$

If $p \geqq 2q$ and *odd*, an important special case arises when

$$\phi \equiv \lambda \,.\, F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \mu \,.\, F_2(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu);$$

for multiplication by λ allows the substitution to be entirely performed at once, giving $S \equiv x \,.\, F_1(x, y, z, s) + y \,.\, F_2(x, y, z, s) = 0$, where $m' = \frac{1}{2}(p+1)$ and S contains the line x, y once, so that this line occurs twice in the residual intersection of the two surfaces. In order that ϕ may not be reducible by the factor λ, it is necessary that some term in F_2 contain ν to the degree $\frac{1}{2}(p-1)$; hence, this case can arise only when $q = \frac{1}{2}(p-1)$, and may be described as

II'. $\phi \equiv \lambda F_1 + \mu F_2, \quad p = 2q + 1, \quad m' = \frac{1}{2}(p+1),$
$f_1 = f_2 = 1, \quad g_1 = 0, \quad \omega \equiv \lambda$ and $f_1'' = f_2'' = 0.$

Geometrically, this case includes only those curves which have q pairs of branches along the double line of Σ; i. e., at each of the q points where the curve ϕ meets the double line in the second sheet, a branch of the curve meets that line at the same point in the first sheet; but, since $p - q = q + 1$, there is one more branch in the first sheet in the neighborhood of the double line than there are branches in the second sheet in the same neighborhood, and, consequently, the introduction of the double line in the second sheet completes the intersection by forming a pair of branches with the single or superfluous branch in the first sheet. This case forms the nearest analogy, when p is odd, to the case of complete intersection, where p is even.

When $\phi = 0$, with $p \geqq 2q$ and odd, does not have the special form just considered, it is evident that the surface S of lowest order is obtained by giving to τ the value unity, leading to the case

II. $\phi \not\equiv \lambda F_1 + \mu F_2, \quad p \geqq 2q, p$ *odd*, $\quad m' = \frac{1}{2}(p+3), \quad f_1 = \frac{1}{2}(p+3) - q,$
$f_2 = \theta_1 + 2, \quad g_1 = 1 - \theta_1, \quad \omega \equiv \lambda^2 . \omega_1,$

and $$f_1' = f_2' = 1, \quad f_1'' = 0, \quad f_2'' = 1, \text{ if } p + 1 = 2q,$$

but $$f_1' = f_2' = \theta_1 + 2, \quad f_1'' = \tfrac{1}{2}(p-1) - q - \theta_1, \quad f_2'' = 0, \text{ if } p + 1 > 2q.$$

If $p < 2q$, τ must have a value greater than zero in the general case, since the condition $p + \tau \geqq 2q$ must be satisfied. The surface S will be of the lowest

possible order when τ is given its lowest possible value $\tau = 2q - p$, leading to the case

$$\text{III.}\quad p \leqq 2q,\quad m' = q + 1,\quad f_1 = 1,\quad f_2 = 2,\quad g_1 = 2q - p,\quad \omega = \lambda^2.\omega_\tau,$$

and $\theta_1 = 0$, since the condition $\theta_1 \leqq \frac{1}{2}(p + \tau) - q - 1$ here becomes impossible of satisfaction by any positive integer. But special cases, in which a surface S of order lower than $q + 1$ can be passed through the curve ϕ, may occur here. If $\phi \equiv \lambda . F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \nu . F_2(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, multiplication by μ allows the equation of S to be obtained at once in the form $S \equiv y . F_1(x, y, z, s) + zF_2(x, y, z, s) = 0$, where $m' = \frac{1}{2}(p + 1)$ and the generator $\mu = 0$ is known to belong to the residual intersection. In order that the equation as given may not be reducible by λ, it is necessary that some term of F_2 involve the variable ν to the degree $\frac{1}{2}(p-1)$; consequently, the given curve must here have $q = \frac{1}{2}(p+1)$. Therefore, $m' = \frac{1}{2}(p+1) = q$; and, since $\frac{3}{2}(p+1) - 1 = \frac{3}{2}(p + \frac{1}{3}) = p + \frac{1}{2}(p + 1) = p + q = m'$, the single generator $\mu = 0$ constitutes the entire residual intersection. This case may be classified thus:

$$\text{III}'.\quad \phi \equiv \lambda F_1 + \nu F_2,\quad p = 2q - 1,\quad m' = q,\quad f_1 = f_2 = 0,\quad \overline{g_1} = 1,\quad \omega \equiv \mu,$$

if $\overline{g_1}$ refer to the particular generator $\mu = 0$. Geometrically, this case includes only those curves which have $q - 1$ pairs of branches along the double line of Σ and an extra branch in the second sheet which meets the double line at the point $\mu = 0$; hence, any curve ϕ, which has pairs of branches without superfluous branches wherever it meets the double line, save that at one point of the double line occurs a superfluous or single branch lying in the second sheet in that neighborhood, can be cut from Σ by a surface S of order q, which surface is found by the method given, after first making such a change of coördinates, if necessary, as shall bring the equation of the generator at the point in question into the form $\mu = 0$. Clearly, this point cannot lie at the pinch-point.

Still another special case can occur here. If ϕ be written in the form

$$\phi \equiv V_p + \lambda . V_{p-1} + \lambda^2 . V_{p-2} + \ldots . + \lambda^\kappa . V_{p-\kappa} + \ldots . + \lambda^{p-1} . V_1 + \lambda^p . V_0 = 0,$$

it has been already seen that $V_p \equiv \mu^{p-q} . \overline{V_q}$, where $\overline{V_q} = 0$ gives the points where the curve ϕ meets the double line in the second sheet of Σ. Hence, the equation $\overline{\overline{V_q}} = 0$, where $\overline{\overline{V_q}}$ denotes what $\overline{V_q}$ becomes when μ and ν have been replaced by λ and $-\mu$ respectively therein, gives the q generators meeting the

double line in the first sheet in the points where the (p, q) meets that line in the second sheet. Here will $\overline{V_q}.\overline{\overline{V_q}} \equiv F(\lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, so that $\omega \equiv \lambda^{p-q}.\overline{V_q}$ gives, by the usual method of substitution, $\omega.\ V_p \equiv y^{p-q}.F(y, z, s)$; if $p > q+1$, the required substitution can be performed throughout every other group of terms of the equation $\omega\phi = 0$ and the equation of the surface S is at once obtained. In this case $m' = p$, and the residual intersection is entirely made up of the double line of Σ occurring $p - q$ times in the first sheet and $p - q + \theta_1'$ times in the second sheet of Σ, together with the $q - \theta_1'$ generators in the first sheet of Σ introduced by $\overline{\overline{V_q}} = 0$; here θ_1' denotes the number of times λ occurs as a factor in all the terms of $\overline{\overline{V_q}} = 0$, and is, geometrically, the order of multiplicity on the curve ϕ of the pinch-point regarded as lying in the second sheet. But the substitution can readily be performed by the usual method in the general case if $\omega \equiv \lambda^2.\overline{\overline{V}}_{q-\theta_1'}$, when $p+q-\theta_1'$ is even, or if $\omega \equiv \lambda^2.(a\lambda + b\mu).\overline{\overline{V}}_{q-\theta_1'}$, when $p+q-\theta_1'$ is odd, provided in both cases that $\theta_1' \leq p - q$; here

$$m' = \tfrac{1}{2}(p+q-\theta_1') + 1 \text{ and } \tfrac{1}{2}(p+q-\theta_1'+1) + 1$$

respectively; that these values be lower than those given in the general case for m', it is necessary that either $\theta_1' > q$ or $\theta_1' > p - q$, neither of which conditions is possible of fulfillment. However, if $p = q$, a surface S of lower order than that given in the general case III. is found, provided that either $V_{p-1} = 0$ or that $\lambda V_{p-1}\overline{\overline{V_p}} \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$; for multiplying by $\omega \equiv \overline{\overline{V_p}}$ in such case makes the substitution possible in every group of terms and gives the resulting surface S, characterized by $m' = p$, $f_1 = f_2 = 0$, $g_1 = p$, $g_2 = 0$; that $\theta_1' = 0$, and that, accordingly, the double line cannot occur in the residual intersection is clear, since the condition that $p = q$ demands that $\nu^q\ (= \nu^p)$ occur in some term of $\phi = 0$, and such a term can be found only in the first group V_p of that equation; this means that no curve (p, q), where $p = q$, can pass through the pinch-point of Σ, for the presence of a term containing ν^p in the equation of the curve forbids that the curve pass through the pinch-point in the second sheet, and the fact that $p - q = 0$ shows that the curve has no points at all on the double line in the first sheet. Geometrically, the multiplication of $\phi = 0$ by $\overline{\overline{V_q}}$ introduces into the locus $\omega\phi = 0$ the generator at every point in the first sheet where the curve $\phi = 0$ meets that line in the second sheet; consequently, the complete intersection of S and Σ has a pair of of branches on the double line wherever that line is met by the curve $\phi = 0$ in

the second sheet; and, if $p = q$, that intersection meets the double line only in the q points, each one of which is a point of multiplicity 2ρ (ρ being a positive integer) on the complete intersection in question. The vanishing of V_{p-1} means that the conic, whose equation is obtained by equating any one of the p factors $a\mu + b\nu$ of V_p to zero, meets the curve ϕ in two consecutive points on the double line of Σ; or, in other words, the curve ϕ has in this case the direction of the conic $a\mu + b\nu = 0$ at every point $\lambda = 0$, $\mu/\nu = -b/a$, where the curve meets the double line. Since b can never take the value zero here, this condition demands that the curve shall not have the direction of the double line at any point where it meets that line. The alternative condition for V_{p-1}, viz., that $\lambda . V_{p-1} . \overline{\overline{V}}_p \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, since V_{p-1} does not involve λ, and since $\overline{\overline{V}}_p$ does not contain λ as a factor, requires V_{p-1} to contain $p-1$ of the p factors which make up V_p. Therefore, at $p-1$ of the p points where the curve ϕ meets the double line in the second sheet must it be tangent to a conic whose equation is of the form $a\mu + b\nu = 0$, where $b \neq 0$; and it is not difficult to see that the tangent at the remaining point cannot be the double line itself. Therefore, in this case also, the curve ϕ cannot have the direction of the double line at any of the points where it meets that line. This very special case may be characterized thus:

$$\text{III}''. \quad \phi \equiv V_p + \lambda . V_{p-1} + \lambda^2 . W_{p-2}, \quad \lambda . V_{p-1} . \overline{\overline{V}}_p \ F \equiv (\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu),$$
$$p = q, \quad m' = p, \quad f_1 = f_2 = 0, \quad \overline{g}_1 = p, \omega \equiv V_p,$$

where $\overline{g}_1$ refers to the particular generators in the first sheet given by the equation $\overline{\overline{V}}_p = 0$.*

Collecting the results thus far obtained for the order of the surface S and the nature and composition of the residual intersection—

1). *When S is a proper surface of any order whatever not less than $\tfrac{1}{2}p + 1$, cutting the curve (p, q) from Σ,*

$$m' = \tfrac{1}{2}(p + \tau) + 1, \quad f_1 = \tfrac{1}{2}(p + \tau) - q + 1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1.$$

2). *When S is the surface, or one of the surfaces, of the lowest possible order cutting the curve (p, q) from Σ,*

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* The case where $\phi \equiv F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \lambda^2 . W_{p-2} = 0$, since λ cannot be a factor of all the terms, can be written $\phi \equiv F_1(\mu\nu, \mu^2 - \lambda\nu) + \lambda^2 . W_{p-2} = 0$, and presents no new result. For, in order that the required substitution be at once possible, the last group of terms must have $q \leqq \tfrac{1}{2}p - 1$, or else the case must reduce itself to that of complete intersection; but the first group demands that $q = \tfrac{1}{2}p$; consequently, this case offers nothing new.

I. $p \geqq 2q$, p *even*, $m' = \frac{1}{2}p + 1$, $f_1 = \frac{1}{2}p - q + 1$, $f_2 = 2$,
$g_1 = 0$, from $\omega \equiv \lambda^2$;

II. $p \geqq 2q$, p *odd*, $m' = \frac{1}{2}(p+3)$, $f_1 = \frac{1}{2}(p+3) - q$, $f_2 = 2 + \theta_1$,
$g_1 = 1 - \theta_1$, from $\omega \equiv \lambda^2 . \omega_1$;

III. $p \leqq 2q$, $m' = q + 1$, $f_1 = 1$, $f_2 = 2$, $g_1 = 2q - p$, from $\omega = \lambda^2 . \omega_{2q-p}$;

unless a surface S of still lower order can be found from the occurrence of one of the following special cases:

I$'$. $p = 2q$, $\phi \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $m' = \frac{1}{2}p$, $f = g = 0$, $\omega \equiv 1$.

II$'$. $p = 2q + 1$, $\phi \equiv \lambda . F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \mu . F_2(\text{id.})$,
$m' = \frac{1}{2}(p+1)$, $f_1 = f_2 = 1$, $g_1 = 0$, $\overline{g_1} = 0$, from $\omega \equiv \lambda$.

III$'$. $p = 2q - 1$, $\phi \equiv \lambda . F_1(\lambda^2, \mu\nu, \mu^2 - \lambda\nu) + \nu . F_2(\text{id.})$, $m' = q$,
$f_1 = f_2 = 0$, $g_1 = 0$, $\overline{g_1} = 1$, from $\omega \equiv \mu$.

III$''$. $p = q$, $\phi \equiv V_p + \lambda . V_{p-1} + \lambda^2 . W_{p-2}$,
$\lambda . V_{p-1} . \overline{\overline{V}}_p \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $m' = p$, $f_1 = f_2 = 0$, $\overline{g_1} = p$, from $\omega \equiv \overline{\overline{V}}_p$.

g_1 here refers to generators which may be chosen, in general, at will, and can be wholly or in part taken over into f_2 as g_2; while $\overline{g_1}$ refers to generators which are determined by the given curve and can none of them be transferred to f_2 or otherwise changed in any way without raising the order of S. This distinction between g_1 and $\overline{g_1}$ will be heeded in the farther use of the terms.

Under 1) are included cases I., II., and III., but not the special cases I.$'$, II.$'$, III.$'$, and III.$''$.

It is easy to find, in any of the above cases, the relation, as regards contact, of the surfaces S and Σ along the double line in either sheet of the latter. Thus it is clear that three cases arise, viz.:

a). If $f_1 = f_2$, then $f_1' = f_2' = f_1 = f_2$, and $f_1'' = f_2'' = 0$.

b). If $f_1 > f_2$, then $f_1' = f_2' = f_2$, $f_1'' = f_1 - f_2$, and $f_2'' = 0$.

c). If $f_1 < f_2$, then $f_1' = f_2' = f_1$, $f_2'' = f_2 - f_1$, and $f_1'' = 0$.

The only curves on Σ which do not meet the generators at all and, consequently, are of the species $(p, 0)$ are the generators themselves. Since these are not proper curves for any value of p greater than unity, all curves (p, q), having $q = 0$ and $p > 1$, will be omitted in the subsequent treatment of the curves on Σ.

The following Table 1 gives, in accordance with the preceding considerations, the possible cases of surfaces S of the lowest possible orders for curves of orders 1–12 on Σ. It will be noticed that each general case, where $p = 2q$, may be

regarded as belonging to both Case I. and Case III.; each $(2q, q)$ is thus entered in the table.

TABLE 1.

m	Species of Curve	Case	m'	$f_1' \atop f_2'$	f_1''	f_2''	g_1	$\bar{g}_1$	θ_1
1	(1, 0)	II″.	1	1	0	0	0	0	0
2	(1, 1)	III.	2	1	0	1	1	0	0
		III′.	1	0	0	0	0	1	0
		III″.	1	0	0	0	0	1	0
3	(2, 1)	{I. III.}	2	1	0	1	0	0	0
		I′.	1	0	0	0	0	0	0
4	(3, 1)	II.	3	2	0	0	1	0	0
		II′.	2	1	0	0	0	0	0
	(2, 2)	III.	3	1	0	1	2	0	0
		III′.	2	0	0	0	0	2	0
5	(4, 1)	I.	3	2	0	0	0	0	0
	(3, 2)	III.	3	1	0	1	1	0	0
		III′.	2	0	0	0	0	1	0
6	(5, 1)	II.	4	2	1	0	1	0	0
				3	0	0	0	0	1
	(4, 2)	{I. III.}	3	1	0	1	0	0	0
		I′.	2	0	0	0	0	0	0
	(3, 3)	III.	4	1	0	1	3	0	0
		III″.	3	0	0	0	0	3	0
7	(6, 1)	I.	4	2	1	0	0	0	0
	(5, 2)	II.	4	2	0	0	1	0	0
		II′.	3	1	0	0	0	0	0
	(4, 3)	III.	4	1	0	1	2	0	0
8	(7, 1)	II.	5	2	2	0	1	0	0
				3	1	0	0	0	1
	(6, 2)	I.	4	2	0	0	0	0	0
	(5, 3)	III.	4	1	0	1	1	0	0
		III′.	3	0	0	0	0	1	0
	(4, 4)	III.	5	1	0	1	4	0	0
		III″.	4	0	0	0	0	4	0
9	(8, 1)	I.	5	2	2	0	0	0	0
	(7, 2)	II.	5	2	1	0	1	0	0
				3	0	0	0	0	1
	(6, 3)	{I. III.}	4	1	0	1	0	0	0
		I′.	3	0	0	0	0	0	0
	(5, 4)	III.	5	1	0	1	3	0	0
10	(9, 1)	II.	6	2	3	0	1	0	0
				3	2	0	0	0	1
	(8, 2)	I.	5	2	1	0	0	0	0
	(7, 3)	II.	5	2	0	0	1	0	0
		II′.	4	1	0	0	0	0	0
	(6, 4)	III.	5	1	0	1	2	0	0
	(5, 5)	III.	6	1	0	1	5	0	0
		III″.	5	0	0	0	0	5	0
11	(10, 1)	I.	6	2	3	0	0	0	0
	(9, 2)	II.	6	2	2	0	1	0	0
				3	1	0	0	0	1
	(8, 3)	I.	5	2	0	0	0	0	0
	(7, 4)	III.	5	1	0	1	1	0	0
		III′.	4	0	0	0	0	1	0
	(6, 5)	III.	6	1	0	1	4	0	0
12	(11, 1)	II.	7	2	4	0	1	0	0
				3	3	0	0	0	1
	(10, 2)	I.	6	2	2	0	0	0	0
	(9, 3)	II.	6	2	1	0	1	0	0
				3	0	0	0	0	1
	(8, 4)	{I. III.}	5	1	0	1	0	0	0
		I′.	4	0	0	0	0	0	0
	(7, 5)	III.	6	1	0	1	3	0	0
	(6, 6)	III.	7	1	0	1	6	0	0
		III″.	6	0	0	0	0	6	0
13	(12, 1)	I.	7	2	4	0	0	0	0
	(11, 2)	II.	7	2	3	0	1	0	0
				3	2	0	0	0	1
	(10, 3)	I.	6	2	1	0	0	0	0
	(9, 4)	II.	6	2	0	0	1	0	0
		II′.	5	1	0	0	0	0	0
	(8, 5)	III.	6	1	0	1	2	0	0

Since it is only in the cases I'., III'., and III''. that a surface S is found which does not have the double line of Σ for a part of its intersection with that surface, it is clear that only in those three cases are restricted systems of equations for the curves in question given by the methods here followed. In the case I'., the intersection is complete; consequently, $S = 0$ and $\Sigma = 0$ form the simplest possible restricted system of equations for the representation of the curve. In the case of a curve coming under III'., a restricted system is formed by $S_1 = 0$, $S_2 = 0$ and $\Sigma = 0$, where S_1 is the surface defined by III'. and S_2 is any one of those defined by the general case with the value $\frac{1}{2}(p+1)$ inserted for q and the condition imposed that g_1 shall not contain the generator given by $\mu = 0$; thus the surface S_1 will be characterized by having

$$m' = p, \quad f_1 = f_2 = 0, \quad \text{and } \overline{g_1} = 1;$$

while S_2 will have

$$m' = \tfrac{1}{2}(p+\tau) + 1, \quad \text{and } f_1 = \tfrac{1}{2}(\tau+1), \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1 \text{ and } \theta_2 = 0.$$

In the simplest case, where $\tau = 2q - p$ and $\theta_1 = 0$, the surface S_2 is defined like the S of Case III. If the curve fall under Case III''., a restricted system is given by $S_1 = 0$, $S_2 = 0$ and $\Sigma = 0$, where S_1 is defined by Case III''. and S_2 is any one of those defined by the general case by putting $p = q$ therein and insuring, by choice of ω_τ, that g_1 contain none of the generators given by $\overline{V_p} = 0$ and represented by $\overline{g_1}$ in III''.; thus S_1 will have

$$m' = p, \quad f_1 = f_2 = 0, \quad \overline{g_1} = p;$$

and S_2 will be characterized by

$$m' = \tfrac{1}{2}(p+\tau) + 1, \quad f_1 = \tfrac{1}{2}(\tau - p) + 1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1,$$

which, for the lowest value, p, that τ can take, gives an S_2 having

$$m' = p + 1, \quad f_1 = 1, \quad f_2 = 2, \quad g_1 = \tau.$$

It is now possible to determine how many species of proper curves (p, q) may result from the intersection of surfaces S of any given order m' with Σ, when it is required that S be a surface of the lowest possible order thus containing the curve, and hence having its residual intersection made up entirely of the double line and generators of Σ. If the intersection be of such nature as to

give any one of the eight cases,

1). $f_1 = f_2 = g_1 = g_1 = 0$;
2). $f_1 = f_2 = g_1 = 0$, $\overline{g_1} = 1$, when $m' \geqq 2$;
3). $f_1 = f_2 = g_1 = 0$, $\overline{g_1} = m'$;
4). $f_1 = f_2 = 1$, $g_1 = \overline{g_1} = 0$;
5). $f_1 = 1$, $f_2 = 2$, $g_1 = \rho$, $\overline{g_1} = 0$, where $2 \leqq \rho \leqq m' - 1$;
6). $f_1 = \rho$, $f_2 = 2$, $g_1 = \overline{g_1} = 0$, where $1 \leqq \rho \leqq m' - 1$;
7). $f_1 = \rho$, $f_2 = 2$, $g_1 = 1$, $\overline{g_1} = 0$, where $1 \leqq \rho \leqq m' - 1$;
8). $f_1 = \rho$, $f_2 = 3$, $g_1 = \overline{g_1} = 0$, where $3 \leqq \rho \leqq m' - 1$;

and if the residual of the intersection is a proper curve, then is S, in general, a surface of the lowest possible order that can be passed through the residual curve, Σ being excepted when $m' \geqq 4$. The lower limit of the order of the curves (p, q) here occurring is evidently $m = 2(m' - 1)$, and the species of the curve is given by the following formulas, which are arranged and numbered to agree with the cases of page 206:

I. $f_1 = \rho$, $f_2 = 2$, $g_1 = \overline{g_1} = 0$, $p = 2(m' - 1)$, $q = m' - \rho$,
$1 \leqq \rho \leqq m' - 1$, $p \geqq 2q$, and *even*; from 6) above.

II. $f_1 = \rho$, $f_2 = 2$, $g_1 = 1$, $\overline{g_1} = 0$, $p = 2m' - 3$, $q = m' - \rho$,
$2 \leqq \rho \leqq m' - 1$, $p \geqq 2q$ and *odd*, $\theta_1 = 0$; from 7) above.

$f_1 = \rho$, $f_2 = 3$, $g_1 = \overline{g_1} = 0$, $p = 2m' - 3$, $q = m' - \rho$,
$3 \leqq \rho \leqq m' - 1$, $p \geqq 2q$ and *odd*, $\theta_1 = 1$; from 8) above.

III. $f_1 = 1$, $f_2 = 2$, $g_1 = \rho$, $\overline{g_1} = 0$, $p = 2(m' - 1) - \rho$,
$q = m' - 1$, $1 \leqq \rho \leqq m' - 1$, $p \leqq 2q$; from 5) above.

I′. $f_1 = f_2 = g_1 = \overline{g_1} = 0$, $p = 2m'$, $q = m'$, $p = 2q$; from 1) above.

II′. $f_1 = 1$, $f_2 = 1$, $g_1 = \overline{g_1} = 0$, $p = 2m' - 1$, $q = m' - 1$,
$p = 2q + 1$; from 4) above.

III′. $f_1 = f_2 = 0$, $g_1 = 0$, $\overline{g_1} = 1$, $p = 2m' - 1$, $q = m'$, $p = 2q - 1$;
from 2) above.

III″. $f_1 = f_2 = 0$, $g_1 = 0$, $\overline{g_1} = m'$, $p = m'$, $q = m'$, $p = q$; from 3) above.

Cases III′. and III″. become identical when $m' = 1$, but for no other values of m' will any repetitions occur from these formulas.

The determination of the number of distinct species of curves, as well as

the number of their different orders, which can be cut from Σ by surfaces of any order m' under the given conditions, can now be made. Arranged according to the above cases, the number of distinct species of curve for any value of m' comes out thus:

I. $p=2m'-2$, $q=m'-1$, $m'-2$, ..., 1, making $m'-1$ species.
II. $p=2m'-3$, $q=m'-2$, $m'-3$, ..., 1, making $m'-2$ species.
III. $p=2m'-3$, $2m'-4$, ..., $m'-1$, $q=m'-1$, making $m'-1$ species.
I'. $p=2m'$, $q=m'$, a single species.
II'. $p=2m'-1$, $q=m'-1$, a single species.
III'. $p=2m'-1$, $q=m'$, a single species.
III''. $p=m'$, $q=m'$, a single species.

Consequently, all together, curves (p, q) of

$$m'-1+m'-2+m'-1+4=3m'$$

distinct species are found. Of different orders, all from $2m'-2$ to $3m'$ occur, making a total of $m'+3$, so long as $m'\geqq 2$; when $m'=1$, this result must be diminished by unity, since no curve of the lowest order, $2m'-2$, then exists. Hence, the following

THEOREM: *Surfaces S of any given order $m'\geqq 2$, can cut from Σ curves (p, q) of $3m'$ distinct species and of $m'+3$ different orders, where S is a surface of the lowest possible order containing the curve (except Σ, if $m'\geqq 4$) and, therefore, has its residual intersection with Σ made up entirely of generators of the latter surface.*

If the curve (p, q) have $q=1$, a plane through the double line of Σ will contain only one point of the curve not lying on that line. Such a curve is unicursal; the surfaces S, of order $m'\geqq 2$, can then, under the given conditions, cut from Σ two unicursal curves for each value of m'; these two curves will be of the species $(2m'-2, 1)$ and $(2m'-3, 1)$ and of the orders $2m'-1$ and $2m'-2$ respectively. When $m'=1$, the unicursal curves found in the same way are of the species $(1, 0)$ and $(1, 1)$ and of the orders 1 and 2 respectively

Results for the lower values of m', in accordance with the formulas stated, are given in Table 2. In this table, as in some of the preceding formulas, curves (p, q), where $p=2q$, which might be given under both Case I. and Case III., are inserted as occurring only in Case I.

TABLE 2.

m'	Case	Species of Curve	m	f'_1 f'_2	f''_1	f''_2	g_1	$\overline{g_1}$	θ_1
1	I'.	(2, 1)	3	0	0	0	0	0	0
	II'.	(1, 0)	1	1	0	0	0	0	0
	{III'. III''.}	(1, 1)	2	0	0	0	0	1	0
2	I.	(2, 1)	3	1	0	1	0	0	0
	III.	(1, 1)	2	1	0	1	1	0	0
	I'.	(4, 2)	6	0	0	0	0	0	0
	II'.	(3, 1)	4	1	0	0	0	0	0
	III'	(3, 2)	5	0	0	0	0	1	0
	III''.	(2, 2)	4	0	0	0	0	2	0
3	I.	(4, 2)	6	1	0	1	0	0	0
		(4, 1)	5	2	0	0	0	0	0
	II.	(3, 1)	4	2	0	0	1	0	0
	III.	(3, 2)	5	1	0	1	1	0	0
		(2, 2)	4	1	0	1	2	0	0
	I'.	(6, 3)	9	0	0	0	0	0	0
	II'.	(5, 2)	7	1	0	0	0	0	0
	III'.	(5, 3)	8	0	0	0	0	1	0
	III''.	(3, 3)	6	0	0	0	0	3	0
4	I.	(6, 3)	9	1	0	1	0	0	0
		(6, 2)	8	2	0	0	0	0	0
		(6, 1)	7	2	1	0	0	0	0
	II.	(5, 2)	7	2	0	0	1	0	0
		(5, 1)	6	2	1	0	1	0	0
				3	0	0	0	0	1
	III.	(5, 3)	8	1	0	1	1	0	0
		(4, 3)	7	1	0	1	2	0	0
		(3, 3)	6	1	0	1	3	0	0
	I'.	(8, 4)	12	0	0	0	0	0	0
	II'.	(7, 3)	10	1	0	0	0	0	0
	III'.	(7, 4)	11	0	0	0	0	1	0
	III''.	(4, 4)	8	0	0	0	0	4	0
5	I.	(8, 4)	12	1	0	1	0	0	0
		(8, 3)	11	2	0	0	0	0	0
		(8, 2)	10	2	1	0	0	0	0
		(8, 1)	9	2	2	0	0	0	0
	II.	(7, 3)	10	2	0	0	1	0	0
		(7, 2)	9	2	1	0	1	0	0
				3	0	0	0	0	1
		(7, 1)	8	2	2	0	1	0	0
				3	1	0	0	0	1
	III.	(7, 4)	11	1	0	1	1	0	0
		(6, 4)	10	1	0	1	2	0	0
		(5, 4)	9	1	0	1	3	0	0
		(4, 4)	8	1	0	1	4	0	0
	I'.	(10, 5)	15	0	0	0	0	0	0
	II'.	(9, 4)	13	1	0	0	0	0	0
	III'.	(9, 5)	14	0	0	0	0	1	0
	III''.	(5, 5)	10	0	0	0	0	5	0
6	I.	(10, 5)	15	1	0	1	0	0	0
		(10, 4)	14	2	0	0	0	0	0
		(10, 3)	13	2	1	0	0	0	0
		(10, 2)	12	2	2	0	0	0	0
		(10, 1)	11	2	3	0	0	0	0
	II.	(9, 4)	13	2	0	0	1	0	0
		(9, 3)	12	2	1	0	1	0	0
				3	0	0	0	0	1
		(9, 2)	11	2	2	0	1	0	0
				3	1	0	0	0	1
		(9, 1)	10	2	3	0	1	0	0
				3	2	0	0	0	1
	III.	(9, 5)	14	1	0	1	1	0	0
		(8, 5)	13	1	0	1	2	0	0
		(7, 5)	12	1	0	1	3	0	0
		(6, 5)	11	1	0	1	4	0	0
		(5, 5)	10	1	0	1	5	0	0
	I'.	(12, 6)	18	0	0	0	0	0	0
	II'.	(11, 5)	16	1	0	0	0	0	0
	III'.	(11, 6)	17	0	0	0	0	1	0
	III''.	(6, 6)	12	0	0	0	0	6	0
7	I.	(12, 6)	18	1	0	1	0	0	0

V.—*Singularities of the Curves* (p, q) *on* Σ *in Terms of* p *and* q.

It is now possible to find the singularities of any given curve on Σ by making use of the knowledge of the nature of the residual intersection of Σ with the surface S, as that surface has been defined in the preceding pages.

Let the characteristics of the developable surface, having the curve in question for its edge of regression, and of the cone standing upon that curve be denoted, in accordance with the usage of Cayley and Salmon,* by the letters $m, n, r, \alpha, \beta, x, y, g, h$ and H.

As already seen, $m = p + q$. It is, in general, possible that the curve of intersection of S and Σ shall contain no cusps arising from the occurrence of stationary contact between these two surfaces; hence, for one such curve, at least, it may be assumed that β has its lowest value, zero. For the third singularity of the given curve, the rank, r, can be found in the following manner:

Given two surfaces, whose equations are $S = 0$ and $\Sigma = 0$, the condition that a tangent to their curve of intersection meet the arbitrary line represented by the equations

$$a_1x + b_1y + c_1z + d_1s = 0 \text{ and } a_2x + b_2y + c_2z + d_2s = 0 \text{ is that}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} & \frac{\partial S}{\partial z} & \frac{\partial S}{\partial s} \\ \frac{\partial \Sigma}{\partial x} & \frac{\partial \Sigma}{\partial y} & \frac{\partial \Sigma}{\partial z} & \frac{\partial \Sigma}{\partial s} \end{vmatrix} = 0,$$

which may be denoted by $\Delta = 0$, where the surface Δ represents the locus of points, the intersections of whose polar planes with respect to S and Σ meet the arbitrary line. If the curve in question be the complete intersection of S and Σ, the rank desired will be the number of points common to S, Σ and Δ; i. e., the product of the orders of those three surfaces, subject to a reduction for the multiple points of the curve. But, if the curve be taken as the partial intersection of S and Σ, a further reduction of that product is necessary in order to obtain r.†

The surface S has been defined, in the general case, when so determined as

* Salmon, "Geometry of Three Dimensions," pp. 291–293.

† Salmon, "Geometry of Three Dimensions," p. 308.

to contain the curve (p, q) and have the residual of its intersection with Σ made up entirely of straight lines, by the formulas (cf. p. 206),

$$m' = \tfrac{1}{2}(p+\tau)+1,\quad f_1 = \tfrac{1}{2}(p+\tau) - q + 1,\quad f_2 = 2,\quad g_1 = \tau,\ \text{when}\ \theta_1 = 0.$$

The surface Σ is known to be of order three and to have a double line. Therefore, if M', F_1, F_2 and G_1 represent, in regard to Δ and Σ, what m', f_1, f_2, and g_1 respectively, denote with reference to S and Σ, it is clear that the surface Δ will have

$$M' = \tfrac{1}{2}(p+\tau)+2,\quad F_1 = \tfrac{1}{2}(p+\tau) - q + 1,\quad F_2 = 2$$

in all cases. As for G_1, it is evident that its value depends on the choice of ω_τ; thus, if ω_τ be so chosen as to represent τ different generators in the first sheet of Σ, it follows that $G_1 = 0$, while if ω_τ include any such generator more than once, then will that generator occur on Δ, so that in such case $G_1 > 0$. In general, if all generators entering more than once in ω_τ be ρ in number and contribute a total order of γ as their component of g_1, where $2\rho \leqq \gamma \leqq \tau$, then will Δ contain ρ generators of Σ and have $G_1 = \gamma - \rho$; for any generator in the first sheet of Σ, which occurs ε times in the intersection of S and Σ, will occur $\varepsilon - 1$ times in the intersection of Δ and Σ.

The curve (p, q) meets the surface Δ, in general, in

$$m \,.\, M' = (p+q)[\tfrac{1}{2}(p+\tau)+2]$$

points, which number must be subjected to reductions in two ways in order to obtain r. First, since the lines common to Δ and Σ are multiple to an order at least as great as two on one or both the surfaces S and Σ, it follows that the polar plane or planes with respect to S and Σ for all points on these lines are one or both indefinite; consequently, the points of intersection of the curve (p, q) with such multiple lines of Δ should be rejected from the above number; this demands a reduction, since (p, q) has $p - q$ points on the double line in the first sheet of Σ, q points on that line in the second sheet of Σ, and q points on every generator in the first sheet of Σ, amounting to

$$(p-q)\,.\,F_1 + qF_2 + qG_1 = (p-q)[\tfrac{1}{2}(p+\tau) - q + 1] + 2q + q\,.(\gamma - \rho).$$

Secondly, wherever the curve (p, q) meets a line common to S and Σ, these two surfaces have contact, since an element of the curve and an element of the common line determine there a plane tangent to a sheet of either surface. Every such plane meets the arbitrary line in question, but such points must be rejected

from the above number, since, in general, the tangent to the curve there will not intersect the arbitrary line. This demands a reduction by p for the points where the curve meets the double line, by ρq for the points on the ρ generators occurring on Δ, and by $(\tau - \gamma) . q$ for the points on the $\tau - \gamma$ generators occurring singly in the residual intersection of S and Σ. Therefore, the second reduction amounts to

$$p + (\rho + \tau - \gamma) . q.$$

Hence, if no further condition of contact between S and Σ be imposed, it is obtained that

$$\begin{aligned} r &= m . M' - (p - q) . F_1 - q . F_2 - q . G_1 - p - (\rho + \tau - \gamma) . q \\ &= p . (M' - F_1 - 1) + q . (M' + F_1 - F_2 - \tau) \\ &= q . (2p - q + 1). \end{aligned}$$

And if it be further required that S and Σ have ordinary contact at H points and stationary contact at β points, the formula for the rank of the given curve (p, q) is

$$r = q . (2p - q + 1) - 2H - 3\beta.$$

Double points resulting from the intersection on the double line of Σ of a pair of branches of the curve (p, q), the branches lying one in either sheet in that neighborhood, are among the singularities included in h and not in H; for, while they are actual double points of the curve, regarded by itself, they are only apparent double points, regarded from the point of view of the geometry on the surface Σ. If such apparent-actual double points have their number denoted by h_2, where $h = h_1 + h_2$, it is evident that $h_2 \leq p/2$; h_1 will then refer to points which are, from the consideration of the geometry on the surface Σ as well as from the point of view of the curve itself, apparent double points; and H will refer only to those multiple points arising from imposed contact of S with Σ, i. e., actual double points from both points of view. Similarly, a cusp, from the point of view of the curve, arises when two branches, lying one in either sheet in that neighborhood, intersect at the pinch-point; but, regarded from the standpoint of the geometry on the surface Σ, such a singularity is not a cusp and consequently will not be included in β, which represents the number of cusps resulting from imposed stationary contact between S and Σ, i. e., singularities which are cusps from both points of view; such apparent-actual cusps will be included in h_2.

The ordinary Plückerian equations connect the singularities here considered by the formulas

$$\begin{aligned} h &= \tfrac{1}{2}[m(m-1)-r], & x &= \tfrac{1}{2}[r(r-1)-3m-n], \\ n &= 3(r-m)+\beta, & y &= x-(n-m), \\ \alpha &= 2(n-m)+\beta, & g &= \tfrac{1}{2}[n(n-1)-r-3\alpha]. \end{aligned}$$

These equations, with the help of the values of m and r already found, give the following complete set of formulas for the singularities of the curve (p, q) in terms of p, q, H, and β:

$$\begin{aligned} m &= p+q, \\ h &= \tfrac{1}{2}p(p-1)+q(q-1), \\ n &= 3(2pq-q^2-p)-2(3H+4\beta), \\ \alpha &= 4p(3q-2)-2q(3q+1)-3(4H+5\beta), \\ r &= q(2p-q+1)-2H-3\beta, \\ x &= 2pq(pq-q^2+q-2)+\tfrac{1}{2}q(q^3-2q^2+5q-4) \\ &\qquad -\tfrac{1}{2}(4pq-2q^2+2q-1)(2H+3\beta)+\tfrac{1}{2}(2H+3\beta)^2+3H+4\beta, \\ y &= 2pq(pq-q^2+q-5)+\tfrac{1}{2}q(q^3-2q^2+11q-2) \\ &\qquad +4p-\tfrac{1}{2}(4pq-2q^2+2q-1)(2H+3\beta)+\tfrac{1}{2}(2H+3\beta)^2+3(3H+4\beta), \\ g &= \tfrac{9}{2}(2pq-q^2-p)^2-11q(2p-q)+\tfrac{1}{2}(27p+5q) \\ &\qquad -[6(2pq-q^2-p)-7](3H+4\beta)+2(3H+4\beta)^2+H. \end{aligned}$$

If the deficiency of the curve (p, q) be denoted by D, it will be shown later (cf. page 220) that

$$D = p(q-1)-\tfrac{1}{2}q(q+1)+1-H-\beta.$$

The number of varieties of curve of each species (p, q), according to the possible conditions of ordinary and stationary contact between the surfaces S and Σ, is evidently $\frac{1}{2}(D+1)(D+2)$.

It was seen on page 210 that the curves (p, q) having $q=1$, are unicursal; the equation for D above shows that such curves have the deficiency zero, and gives as the condition for zero deficiency when $H=\beta=0$ that q have the value unity.

The singularities of several curves of the lower orders, as computed by the help of the formulas found above, are presented in Table 3.

TABLE 3.

m	Species.	h	D	H	β	r	n	a	x	y	g
1	(1, 0)	0	0	0	0	0	0	0	0	0	0
2	(1, 1)	0	0	0	0	2	0	0	0	0	0
3	(2, 1)	1	0	0	0	4	3	0	0	0	1
4	(3, 1)	3	0	0	0	6	6	4	6	4	6
	(2, 2)	3	0	0	0	6	6	4	6	4	6
5	(4, 1)	6	0	0	0	8	9	8	16	12	20
	(3, 2)	5	1	0	0	10	15	20	30	20	70
			0	1	0	8	9	8	16	12	20
			0	0	1	7	7	5	10	8	10
6	(5, 1)	10	0	0	0	10	12	12	30	24	43
	(4, 2)	8	2	0	0	14	24	36	70	52	125
			1	1	0	12	18	24	48	36	111
			1	0	1	11	16	21	38	28	83
			0	2	0	10	12	12	30	24	43
			0	1	1	9	10	9	22	18	27
			0	0	2	8	8	6	13	15	13
	(3, 3)	9	1	0	0	12	18	24	48	36	111
			0	1	0	10	12	12	30	24	43
			0	0	1	9	10	9	22	18	27
7	(6, 1)	15	0	0	0	12	15	16	48	40	75
	(5, 2)	12	3	0	0	18	33	52	126	100	441
			2	1	0	16	27	40	96	76	283
			2	0	1	15	25	37	82	64	237
			1	2	0	14	21	28	70	56	162
			1	1	1	13	19	25	58	46	127
			1	0	2	12	17	22	47	37	97
			0	3	0	12	15	16	48	40	75
			0	2	1	11	13	13	38	32	53
			0	1	2	10	11	10	29	36	35
			0	0	3	9	9	7	21	19	31
	(4, 3)	12	3	0	0	18	33	52	126	100	441
			2	1	0, etc., repeating the above cases in order.						
8	(7, 1)	21	0	0	0	14	18	20	70	60	116
	(6, 2)	17	4	0	0	22	42	68	198	164	748
			3	1	0	20	36	56	160	132	536
			3	0	1	19	34	53	142	116	472
			2	2	0	18	30	44	126	104	360

m	Species.	*h*	*D*	*H*	β	*r*	*n*	*a*	*x*	*y*	*g*
			2	1	1	17	28	41	110	90	308
			2	0	2	16	26	38	95	77	260
			1	3	0	16	24	32	96	80	220
			1	2	1	15	22	29	82	68	180
			1	1	2	14	20	26	69	57	144
			1	0	3	13	18	23	57	47	112
			0	4	0	14	18	20	70	60	116
			0	3	1	13	16	17	58	50	88
			0	2	2	12	14	14	47	41	64
			0	1	3	11	12	11	37	33	44
			0	0	4	10	10	8	28	26	28
	(5, 3)	16	5	0	0	24	48	80	240	236	1128
			4	1	0	22	42	68	198	164	748
			4	0	1	21	40	65	178	146	672
			3	2	0	20	36	56	160	132	536
			3	1	1	19	34	53	142	116	472
			3	0	2	18	32	50	125	101	412
			2	3	0	18	30	44	126	104	360
			2	2	1	17	28	41	110	90	308
			2	1	2	16	26	38	95	77	260
				0	3	15	24	35	81	65	216
			1	4	0	16	24	32	96	80	220
			1	3	1	15	22	29	82	68	180
			1	2	2	14	20	26	69	57	144
			1	1	3	13	18	23	57	47	112
			1	0	4	12	16	20	46	38	84
			0	5	0	14	18	20	70	60	116
			0	4	1	13	16	17	58	50	88
			0	3	2	12	14	14	47	41	64
			0	2	3	11	12	11	37	33	44
			0	1	4	10	10	8	28	26	28
			0	0	5	9	8	5	20	20	16
	(4, 4)	18	3	0	0	20	36	56	160	132	536
			2	1	0	18	30	44	126	102	310
			2	0	1	17	28	41	110	90	303

VI.—*Geometry on Σ from the Point of View of Plane Curves.*

Any curve (p, q) in the geometry on Σ has as its analogue in plane geometry an entirely definite curve; the equation of the former being $\phi(\lambda, \mu, \nu) = 0$, the equation of the corresponding plane curve is $\phi(x, y, z) = 0$. The order of (p, q) as a curve in space is $p+q$, but the order of the corresponding curve in the plane, and also of the given curve from the point of view of the geometry on Σ, is p. To the $p-q$ points of (p, q), which lie on the double line in the first sheet of Σ corresponds on the plane curve a multiple point, the order of whose multiplicity is $p-q$; the line on Σ is given by $\lambda/\nu = \mu/\nu = 0$, and the point in the plane by $x=0$, $y=0$. To this fact that to the $p-q$ points on a line on Σ corresponds a $(p-q)$-tuple point in the plane are due the chief differences between the geometry on Σ and plane geometry.

1. *Intersections of Two Curves* (p, q) *and* (p', q').

Two plane curves of orders p and p' intersect in pp' points. But since the two curves (p, q) and (p', q') will, in general, have no common points on the double line in the first sheet of Σ to correspond to the points of intersection of the two analogous plane curves at the multiple point $x=0$, $y=0$, the number of intersections of the curves (p, q) and (p', q') will be less than pp' by the number of points of intersection of the two corresponding plane curves at their multiple point in question, i. e, pp' must be diminished by $(p-q)(p'-q')$. Hence, if (p, q) and (p', q') have no points of intersection on the double line in the first sheet of Σ, they will intersect in $pp'-(p-q)(p'-q')$ points. If, however, the two curves (p, q) and (p', q') have a common point or points on the double line in the first sheet of Σ, further consideration is necessary. If the equations of the two curves are

$$\phi(\lambda, \mu, \nu) = U_p + \nu \,.\, U_{p-1} + \cdots + \nu^{\kappa}.U_{p-\kappa} + \cdots + \nu^{q-1}.U_{p-q+1} + \nu^{q}.U_{p-q} = 0 \text{ and}$$

$$\phi'(\lambda, \mu, \nu) = U'_{p'} + \nu \,.\, U'_{p'-1} + \cdots + \nu^{\kappa'}.U'_{p'-\kappa'} + \cdots + \nu^{q'-1}.U'_{p'-q'+1} + \nu^{q'}.U'_{p'-q'} = 0,$$

the condition that θ points of the line in question be common to the two curves, or that the order of intersection of the two curves on that line be θ, is that U_{p-q} and $U'_{p'-q'}$ have θ linear factors common. If the equations of the two corresponding plane curves be now obtained by replacing λ, μ, and ν by x, y, and z

respectively in the equations of (p, q) and (p', q'), giving $\phi(x, y, z) = 0$ and $\phi'(x, y, z) = 0$, the condition for θ intersections on the double line in the first sheet of Σ in the former case becomes in the latter case the condition that θ branches of the one curve be tangent to θ branches of the other curve at the point $x = 0$, $y = 0$. Hence, the two plane curves will have $pp' - (p - q)(p' - q') - \theta$ intersections apart from the multiple point in question. Similarly, the given curves (p, q) and (p', q') will have $pp' - (p - q)(p' - q') - \theta$ intersections apart from the double line in the first sheet of Σ. This number, together with the θ intersections lying on the double line in the first sheet of Σ, gives a total of $pp' - (p - q)(p' - q')$ intersections in this case also. Therefore, from the point of view of the geometry on Σ is established for all cases the

THEOREM.—*The two curves (p, q) and (p', q') on Σ have $pq' + q(p' - q')$ intersections.** θ of these intersections will lie on the double line in the first sheet of Σ when, and only when, the two corresponding plane curves have θ branches of the one tangent to θ branches of the other at the multiple point $x = 0$, $y = 0$. If either $p = q$ or $p' = q'$, then will not only θ have the value zero, but, furthermore, the number of intersections of (p, q) and (p', q') will be the same as the number of intersections of the two corresponding plane curves.

The above formula includes only those intersections which occur as such in the geometry on the surface Σ and does not take account of cases where a branch or branches of each curve meets the double line of Σ at the same point, the branch or branches of the one curve lying in the one sheet and the branch or branches of the other curve lying in the other sheet of Σ in that neighborhood; but the intersections thus occasioned, although to be regarded as only apparent from the standpoint of the geometry on Σ, are actual from the point of view of the curves (p, q) and (p', q') as curves in space. If the number of such intersections be denoted by θ_0, it is evident that $0 \leqq \theta_0 \leqq (p - q)q' + (p' - q')q$, and that $pq' + p'q - qq' + \theta_0$ is the number of intersections possible to any two curves (p, q) and (p', q') regarded as curves in space; evidently this number can be greater than, equal to, or less than the number of intersections of the corresponding curves in the plane; it will necessarily be the same as in the case of the analogous plane curves when both $p = q$ and $p' = q'$.

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* This agrees with the formula $(a_\alpha, b_\beta) = a\beta + b\alpha - 3\alpha\beta$ given by Professor Story, "On the Number of Intersections of Curves Traced on a Scroll of any Order," Johns Hopkins University Circulars, August, 1883.

2. *Double Points on* (p, q).

A plane curve of order p can have at most $\frac{1}{2}(p-1)(p-2)$ double points. A point of the $(p-q)^{\text{th}}$ order of multiplicity counts as equivalent to $\frac{1}{2}(p-q)(p-q-1)$ double points; hence, the plane curve, which is the analogue of the curve (p, q) on Σ, can have a maximum of

$$\tfrac{1}{2}(p-1)(p-2)-\tfrac{1}{2}(p-q)(p-q-1)=p(q-1)-\tfrac{1}{2}q(q+1)+1$$

double points apart from the $(p-q)$-tuple point $x=0$, $y=0$. Consequently, from the point of view of the geometry on Σ, the curve (p, q) can have at most $p(q-1)-\frac{1}{2}q(q+1)+1$ double points; hence, $D=p(q-1)-\frac{1}{2}q(q+1)+1-H-\beta$, as stated on page 215. If a multiple point or multiple points, whose total order of multiplicity is γ, lie on the double line in the first sheet of Σ, then must the branches of the corresponding plane curve have contact of the total order γ at the multiple point $x=0$, $y=0$; evidently, γ must have the value zero when $p=q$; and, furthermore, any curve (p, q), where $p=q$, has the same maximum number of double points as the corresponding plane curve.

The results obtained thus far in this section have to do only with actual double points; i. e., points whose order of multiplicity comes entirely from the intersection of branches lying in the same sheet in the neighborhood of the point in question in each case. But it has been already noticed that apparent-actual multiple points—at most, only apparent from the standpoint of the geometry on Σ, but actual from the point of view of (p, q) as a curve in space—may occur from the intersection on the double line of branches of the curve, some of which branches lie in the one sheet of Σ and another or others lie in the other sheet of Σ in the neighborhood of the point of intersection in question. If no actual multiple points occur on the double line of Σ, there can be at most q such apparent-actual double points on the curve (p, q), if $p \geqq 2q$, and at most $p-q$ such double points on that curve, if $p \leqq 2q$. In general, if β_1 branches intersect at a point of the double line, and all lie in the first sheet of Σ in that neighborhood, and also β_2 branches, all lying in the second sheet of Σ in the neighborhood of the double line, intersect at the same point of that line, the point in question counts as a $\frac{1}{2}(\beta_1+\beta_2)(\beta_1+\beta_2-1)$-tuple point of the curve to which those branches belong, if that curve (p, q) be regarded as a curve in space. The same point has for its order of multiplicity on (p, q), regarded from the standpoint of the geometry on Σ and the corresponding plane curve, only the sum $\frac{1}{2}\beta_1(\beta_1-1)$

$+ \frac{1}{2}\beta_2(\beta_2 - 1)$; hence, the increase in the order of multiplicity of the point, due to the introduction of the apparent-actual multiple points, is

$$\tfrac{1}{2}(\beta_1 + \beta_2)(\beta_1 + \beta_2 - 1) - \tfrac{1}{2}\beta_1(\beta_1 - 1) - \tfrac{1}{2}\beta_2(\beta_2 - 1) = \beta_1\beta_2,$$

which may be as great as $q(p-q)$. Therefore, the maximum sum of the orders of multiplicity of all the multiple points of (p, q), regarded as a curve in space, is

$$p(q-1) - \tfrac{1}{2}q(q+1) + 1 + q(p-q) = p(2q-1) - \tfrac{1}{2}q(3q+1) + 1.$$

So the curve (p, q) can have the sum of the orders of multiplicity of all its multiple points less than, equal to, or greater than the corresponding sum in the case of the analogous plane curve. The above number $p(2q-1) - \frac{1}{2}q(3q+1) + 1$ reduces to $\frac{1}{2}(p-1)(p-2)$, when $p = q$, as is evident geometrically, since no point of the curve (p, q), when $p = q$, lies on the double line in the first sheet of Σ. Any curve (p, q), which has $q = 1$ and, consequently, is unicursal, and, from the point of view of the geometry on Σ, of deficiency zero, can still have an order of multiplicity of all its multiple points together of at most $p - 1$, according as the $p - 1$ points on the double line in the first sheet of Σ lie, in part or even wholly, at the point where that line is met by the curve in question in the second sheet of Σ; thus the quantity $p(2q-1) - \frac{1}{2}q(3q+1) + 1$ reduces to $p - 1$ when q has the value unity.

3. *Determination of the Curve* (p, q).

The equation of the curve (p, q), when written in the form

$$\phi \equiv U_p + \nu \,.\, U_{p-1} + \ldots + \nu^{\kappa}.\, U_{p-\kappa} + \ldots + \nu^{q-1}.\, U_{p-q+1} + \nu^{q}.\, U_{p-q} = 0,$$

is seen to contain, in the general case,

$$p + 1 + p + p - 1 + \ldots + p - q + 2 + p - q + 1$$
$$= pq + p - \tfrac{1}{2}q(q-1) + 1$$

terms; consequently, the curve (p, q) is determined by $p(q+1) - \frac{1}{2}q(q-1)$ points.

The same can be seen also in this way: The general p-thic in the plane is determined by $\frac{1}{2}p(p+3)$ points. The equation above, representing (p, q), has all the terms of the general p-thic save those in the powers of ν beyond ν^q, which are

$$\nu^{q+1}(U_{p-q-1} + \nu \,.\, U_{p-q-2} + \ldots + \nu^{p-q-1}.\, U_0).$$

The number of these terms is

$$p-q+p-q-1+\ldots\ldots+2+1=\tfrac{1}{2}(p-q)(p-q+1).$$

This number of conditions must be subtracted from $\frac{1}{2}p(p+3)$, giving

$$\tfrac{1}{2}p(p+3)-\tfrac{1}{2}(p-q)(p-q+1)=p(q+1)-\tfrac{1}{2}q(q-1)$$

as the number of conditions or points necessary to determine a curve on Σ.

Unicursal curves, having always $q=1$, are each determined by $2p$ points, as the formula shows; and any (p, q), where $p=q$, is seen, from the point of view of its analogue in the plane, as well as from the formula given, to be determined by $\frac{1}{2}p(p+3)$ points.

Table 4 gives the number of points necessary to determine a curve (p, q) of the species indicated, for all curves having $p \leqq 10$.

TABLE 4.

Curve	Points	Curve	Points	Curve	Points	Curve	Points	Curve	Points
(1, 0)	1	(5, 2)	14	(7, 3)	25	(8, 8)	44	(10, 3)	37
(1, 1)	2	(5, 3)	17	(7, 4)	29	(9, 1)	18	(10, 4)	44
(2, 1)	4	(5, 4)	19	(7, 5)	32	(9, 2)	26	(10, 5)	50
(2, 2)	5	(5, 5)	20	(7, 6)	34	(9, 3)	33	(10, 6)	55
(3, 1)	6	(6, 1)	12	(7, 7)	35	(9, 4)	39	(10, 7)	59
(3, 2)	8	(6, 2)	17	(8, 1)	16	(9, 5)	44	(10, 8)	62
(3, 3)	9	(6, 3)	21	(8, 2)	23	(9, 6)	48	(10, 9)	64
(4, 1)	8	(6, 4)	24	(8, 3)	29	(9, 7)	51	(10, 10)	65
(4, 2)	11	(6, 5)	26	(8, 4)	34	(9, 8)	53		
(4, 3)	13	(6, 6)	27	(8, 5)	38	(9, 9)	54		
(4, 4)	14	(7, 1)	14	(8, 6)	41	(10, 1)	20		
(5, 1)	10	(7, 2)	20	(8, 7)	43	(10, 2)	29		

It is evident that here, as in the case of the determination of plane curves by points, the resultant curve will be improper when any one of certain relations exists between the given points. Thus, if more than $p-q$ of the points chosen for the determination of a (p, q) lie on the double line in the first sheet of Σ, the curve must contain that line; and, similarly, if more than q of the $p(q+1)-\frac{1}{2}q(q-1)$ points lie on any generator of Σ, the curve determined must contain that generator; hence, if, in either of these cases, the curve be of order greater than unity, it will be an improper curve containing as a component a straight line. So, in general, when $p(q+1)-\frac{1}{2}q(q-1)$ points for the deter-

mination of a (p, q) are given, since a $(p-p', q-q')$ is determined by $(p-p')(q-q'+1)-\frac{1}{2}(q-q')(q-q'-1)$ points, it follows that when as many as

$$p(q+1)-\tfrac{1}{2}q(q-1)-(p-p')(q-q'+1)+\tfrac{1}{2}(q-q')(q-q'-1)$$
$$=p'(q-q'+1)+q'(p-q)+\tfrac{1}{2}q'(q'+1)$$

of the given points lie on a (p', q'), then the (p, q) is made up of the two curves (p', q') and $(p-p', q-q')$; but it is necessary here that $p' \leqq p$ and $q' \geqq q$, and, consequently, $p-p' \geqq q-q'$. Under these conditions it may be stated that, if, of the $p(q+1)-\frac{1}{2}q(q-1)$ points given for the determination of a (p, q), as many as $p'(q-q'+1)+q'(p-q)+\frac{1}{2}q'(q'+1)$ lie on a (p', q'), the (p, q) will consist of this (p', q'), together with the $(p-p', q-q')$ determined by the $(p-p')(q-q'+1)-\frac{1}{2}(q-q')(q-q'-1)$ remaining points.

Thus, if two of the $2p$ points given for the determination of a unicursal curve $(p, q)_{q=1}$ lie on a generator, the curve consists of that generator, together with the unicursal curve $(p-1, 1)$ determined by the $2(p-1)$ remaining points; and, in general, if $2p-\alpha$ of the $2p$ points given for the determination of a unicursal curve $(p, q)_{q=1}$ lie on a unicursal curve $(p-\alpha, 1)$, the α remaining points determine α generators, which, together with the $(p-\alpha, 1)$, constitute the $(p, q)_{q=1}$ in question. As a particular case, let there be given 50 points for the determination of a (10, 5); if 46 of those 50 points lie on a (6, 5), the 4 remaining points determine a (4, 0), 4 generators, which, with the (6, 5), constitute the (10, 5); or, if 42 of those 50 points lie on a (6, 4), the (10, 5) consists of that (6, 4), together with the (4, 1) determined by the 8 remaining points; or, again, if 39 of those 50 points lie on a (6, 3), the (10, 5) consists of that (6, 3), together with the (4, 2) determined by the 11 remaining points; or, similarly, if 37 of those 50 points lie on a (6, 2), the (10, 5) consists of that (6, 2), together with the (4, 3) determined by the 13 remaining points; if 36 of those 50 points lie on a (6, 1), the (10, 5) consists of that (6, 1), together with the (4, 4) determined by the 14 remaining points; if 48 of those 50 points lie on a (9, 4), the (10, 5) consists of that (9, 4), together with the (1, 1) determined by the 2 remaining points, etc., etc.

A case not in accord with what has been stated in this section occurs when the number of points prescribed by the formula for the determination of the curve is the same as the number of points in which two curves of the species of the

given curve intersect; i. e., when $p(q+1) - \frac{1}{2}q(q-1) = 2pq - q^2$; this demands that $p = \frac{1}{2}\frac{q(q+1)}{q-1}$ and is satisfied by $p = 3$ and $q = 2$ or 3, but by no other values of p and q. Thus, a (3, 2) is determined by 8 points and two (3, 2)'s intersect in 8 points; if $\phi_1 = 0$ and $\phi_2 = 0$ are the equations of two (3, 2)'s which both pass through 7 given points, $\phi_1 + k\phi_2 = 0$ is the equation of a (3, 2) passing through all the 8 intersections of the two given (3, 2)'s; consequently, all (3, 2)'s having 7 points common have in addition an 8th common point; and if the 8 points given for the determination of a (3, 2) are the points of intersection of two (3, 2)'s, the desired curve is not completely determined; for a single infinity of (3, 2)'s can be passed through the 8 given points, and one (3, 2) can be passed through the 8 given points and any 9th point. Similarly, and just as in the case of plane curves of the third order, a single infinity of (3, 3)'s can be passed through the 9 points of intersection of any two (3, 3)'s; and a 10th point is necessary to define a (3, 3) whenever the 9 given points are the points of intersection of two (3, 3)'s. Thus, if 32 of the 50 points given for the determination of a (10, 5) lie on a (7, 3), the (10, 5) consists of that (7, 3) and the (3, 2) determined by the 8 remaining points, if those 8 points are not the points of intersection of two (3, 2)'s; and, similarly, if 41 of the 50 points given for the determination of a (10, 5) lie on a (7, 2), the (10, 5) consists of that (7, 2) and the (3, 3) determined by the 9 remaining points, if those 9 points are not the points of intersection of two (3, 3)'s.

From the corresponding theorems in plane geometry are derived the following for the geometry on Σ:

a). If the two curves (p', q') and (p'', q'') have no multiple points of intersection,

1). Any $(p, q)_{q=p-\kappa}$ which passes through all the points of intersection of $(p', q')_{q'=p'}$ and $(p'', q'')_{q'' \leqq p''}$, whose equations are $\phi' = 0$ and $\phi'' = 0$, can have its equation put in the form $\phi \equiv A\phi' + B\phi'' = 0$, where $A = 0$ and $B = 0$ represent curves of the species $(p - p', q - q' - \kappa)$ and $(p - p'', q - q'' - \kappa)$ respectively. Here must clearly $\kappa \geqq q - q'' - (p - p'')$.

2). Any $(p, q)_{q=p-(p'-q')-(p''-q'')+1}$, which passes through all the points of intersection of $(p', q')_{q'<p'}$ and $(p'', q'')_{q''<p''}$, whose equations are $\phi' = 0$ and $\phi'' = 0$, can have its equation expressed in the form $\phi \equiv A\phi' + B\phi'' = 0$, where $A = 0$ and $B = 0$ represent curves of the species $(p - p', p - p' - (p'' - q'') + 1)$ and $(p - p'', p - p'' - (p' - q') + 1)$ respectively.

b). If the two curves (p', q') and (p'', q'') have an r-tuple point P of the former coincident with an s-tuple point P of the latter,

3). Any $(p, q)_{q=p-\kappa}$, which passes through all the points of intersection of $(p', q')_{q'=p'}$ and $(p'', q'')_{q''\leqq p''}$, whose equations are $\phi' = 0$ and $\phi'' = 0$, will have an $(r + s - 1)$-tuple point at P, and can have its equation expressed in the form $\phi \equiv A\phi' + B\phi'' = 0$, where $A = 0$ represents a curve of the species $(p - p', q - q' - \varkappa)$ having an $(s - 1)$-tuple point at P, and $B = 0$ represents a curve of the species $(p - p'', q - q'' - \varkappa)$ having an $(r - 1)$-tuple point at P. Here evidently $\varkappa \geqq q - q'' - (p - p'')$.

4). Any $(p, q)_{q=p-(p'-q')-(p''-q'')+1}$, which passes through all the points of intersection of $(p', q')_{q'<p'}$ and $(p'', q'')_{q''<p''}$, whose equations are $\phi' = 0$ and $\phi'' = 0$, will have an $(r + s - 1)$-tuple point at P, and can have its equation expressed in the form $\phi \equiv A\phi' + B\phi'' = 0$, where $A = 0$ represents a curve of the species $(p - p', p - p' - (p'' - q'') + 1)$ having an $(s - 1)$-tuple point at P, and $A = 0$ represents a curve of the species $(p - p'', p - p'' - (p' - q') + 1)$ having an $(r - 1)$-tuple point at P.

Many interesting applications of these four theorems are possible. It follows from 1) that, if $pq'' + p''q - qq''$ of the $pq' + p'q - qq'$ intersections of a $(p, q)_{q=p-\kappa}$ and a (p', q') lie on a $(p'', q'')_{q''\leqq p''}$, the remaining points of intersection lie on a $(p - p'', q - q'' - \varkappa)$; or, if pp'' of the $2pq - q^2$ intersections of two (p, q)'s lie on a $(p'', q'')_{q''=p''}$, the remaining points of intersection lie on a $(p - p'', q - q'')$. This means that, if 8 of the 16 points of intersection of two (4, 4)'s lie on a (2, 2), the remaining 8 points also lie on a (2, 2); if 6 of the 9 points of intersection of two (3, 3)'s lie on a (2, 2), the 3 remaining points lie on a (1, 1); and, again, if 17 of the 30 points of intersection of a (6, 5), with a (5, 5) lie on a (3, 2), the 13 remaining points lie also on a (3, 2).

From 2) it follows that, if $pq'' + p''q - qq''$ of the $pq' + p'q - qq'$ points of intersection of a $(p, q)_{q<p}$ and a $(p', q')_{q'<p'}$ lie on a $(p'', q'')_{q''=p''-(p-q)+(p'-q')-1}$, the remaining points of intersection lie on a $(p - p'', p - p'' - (p' - q') + 1)$, where evidently $p' - q' \leqq (p - q) + 1$; and if $pq'' + p''q - qq''$ of the $2pq - q^2$ points of intersection of two $(p, q)_{q<p}$'s lie on a $(p'', q'')_{q''=p''-1}$, the remaining points of intersection lie on a $(p - p'', q - p'' + 1)$. Thus, if 5 of the 8 points of intersection of two (3, 2)'s lie on a (2, 1), the 3 remaining points lie on a (1, 1); and if 21 of the 30 points of intersection of a (6, 4) and a (6, 3) lie on a (4, 3), the 9 remaining points lie on a (2, 1).

Similar applications of Theorems 3) and 4) are readily obtained.

Theorem 1) shows also that any $(p, q)_{q=p}$, which passes through $p(q+1) - \frac{1}{2}q(q-1) - 1$ points of intersection of another $(p, q)_{q=p}$ and a $(p, q')_{q'=p-\kappa}$, contains also the $\frac{1}{2}(p-1)(p-2)$ remaining points of intersection if these do not all lie on a $(p-3, q-3)$. Thus any $(4, 4)$, which passes through 13 points of intersection of another $(4, 4)$ and a $(4, 2)$, contains also the 3 remaining points of intersection, unless those 3 points all lie on a $(1, 1)$; etc., etc.

4. *The Conic* $(1, 1)$ *on* Σ.

The most general linear equation in the three variables, λ, μ and ν, is of the form $a\lambda + b\mu + c\nu = 0$ and represents a curve of the species $(1, 1)$, which may be called a conic $(1, 1)$ on Σ. This curve meets every generator of Σ in one point, has consequently one point on the double line in the second sheet, but has no points on the double line in the first sheet of Σ. If $b = 0$, this conic passes through the point $x = 0$, $y = 0$, $z = 0$ in the second sheet of Σ, and, if $a = b = 0$, it is the conic which, with the generator $y = 0$, $z = 0$, forms the intersection of the plane $z = 0$ with Σ. If $a = 0$ and $b \neq 0$, this conic passes through the point $y = 0, z = 0, s = 0$. If $c = 0$, the curve is no longer a conic but a generator $(1, 0)$; but in this case it may be said that the conic consists of the generator in question and the double line in the first sheet of Σ, since the conic $(1, 1)$ tends to become that composite curve as a limiting case when c is made to approach zero; thus the $(1, 0)$ and the $(0, 1)$ together still constitute a conic $(1, 1)$ when $c = 0$, although the curve is no longer a proper conic.

A conic $(1, 1)$ is evidently determined by two points, neither of which lies on the double line in the first sheet of Σ; nor can both lie on any generator, if the curve is to be proper. The equation of the conic $(1, 1)$ through the points (1) and (2) may be written in the determinant form thus:

$$\begin{vmatrix} \lambda & \mu & \nu \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0.$$

The condition that this conic pass through the pinch-point in the second sheet of Σ is obtained by putting $\lambda_1 = \mu_1 = 0$, which makes (1) become that point; this substitution gives:

$$\begin{vmatrix} \lambda & \mu & \nu \\ 0 & 0 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0,$$

or $\mu_2\lambda - \lambda_2\mu = 0$, for the equation of the conic; but this equation gives only the generator $\lambda/\mu = \lambda_2/\mu_2$ through the given point (2); or, the improper conic in this case may be regarded as made up of this generator and the double line in the first sheet of Σ. This agrees with what was stated earlier, that no $(p, q)_{q=p}$ can pass through the pinch-point, and requires that point to be regarded as lying in the first sheet rather than in both sheets of Σ. If both points, (1) and (2), lie on the same generator, $\lambda_1/\mu_1 = \lambda_2/\mu_2$, the equation becomes $\mu_1\lambda - \lambda_1\mu = 0$; and, if that generator be the double line in the second sheet, $\lambda_1 = \lambda_2 = 0$, the equation becomes $\lambda = 0$; hence, whenever the two points lie on the same generator, the equation reduces to the equation of that generator, and the conic can be regarded as made up of that generator and the double line in the first sheet of Σ. Similarly, if one of the two points, as, e. g., the point (1), be chosen on the double line in the first sheet of Σ, the equation becomes that of the generator through the other point, viz., $\mu_2\lambda - \lambda_2\mu = 0$; and, if both points be chosen on the double line in the first sheet of Σ, the equation becomes $\lambda - \frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2} \cdot \mu = 0$, giving a generator determined by the two points, but containing neither of them so long as those points are distinct; hence the double line in the first sheet must always enter here to constitute, together with the generator found, the improper conic (1, 1) through, and determined by, the two points.

Evidently the coördinates of any point on the conic (1, 1) can be expressed linearly in terms of the coördinates of any two of its points thus:

$$\lambda = s\lambda_1 + t\lambda_2,$$
$$\mu = s\mu_1 + t\mu_2,$$
$$\nu = s\nu_1 + t\nu_2,$$

It has been seen already that two (1, 1)'s intersect in a single point. If the equations of the two conic (1, 1)'s on Σ be $a_1\lambda + b_1\mu + c_1\nu = 0$ and $a_2\lambda + b_2\mu + c_2\nu = 0$, the point of intersection of the two curves has the coördinates

$$\lambda : \mu : \nu = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This point lies on the double line in the second sheet if $b_1/b_2 = c_1/c_2$, and cannot lie on that line in the first sheet of Σ so long as either conic (1, 1) is a proper curve.

In a similar way can all theorems concerning descriptive properties of lines

in the plane be applied to the case of the curves of species $(1, 1)$ on Σ. And in like manner can theorems concerning descriptive properties of curves of any order in the plane be applied to the cases of the corresponding curves on Σ.

5. *Polar and Tangent Curves on* Σ.

If the operator $\lambda \frac{\partial}{\partial \lambda'} + \mu \frac{\partial}{\partial \mu'} + \nu \frac{\partial}{\partial \nu'}$ be denoted by Δ, then may the curve whose equation is $\Delta^k \phi' = 0$ be called the $(p-k)^{\text{th}}$ polar of the point (λ', μ', ν'), or P', with respect to the curve whose equation is $\phi = 0$. If $k \leqq q$, this $(p-k)^{\text{th}}$ polar is a curve of the species (k, k), while, if $k \geqq q$, it is of the species (k, q); consequently, as a curve in space, the $(p-k)^{\text{th}}$ polar is a $2k$-thic or a $(k+q)$-thic, according as $k \leqq q$ or $k \geqq q$, and may be designated as the polar $2k$-thic or polar $(k+q)$-thic, respectively, of the point P' with respect to the curve (p, q). If the point P' be taken on the curve (p, q), the polar $2k$-thic and the polar $(k+q)$-thic become the tangent $2k$-thic and the tangent $(k+q)$-thic respectively; these curves have $k+1$ points lying on the curve (p, q) at the point P' and therefore give, in general, the direction of that curve at the point in question.

Thus, if P' be a point of the first order of multiplicity on (p, q) and do not lie on the double line in the second sheet of Σ, the direction of (p, q) is given most simply by the tangent conic, whose equation is $\Delta\phi' = 0$. And, if P' be a point whose order of multiplicity is k on (p, q) and do not lie on the double line in the second sheet of Σ, the directions of the k branches are, in general, given by the tangent $2k$-thic or $(p+k)$-thic, whose equation is $\Delta^k\phi' = 0$; but multiplicity must here be regarded as referring only to the intersections of branches all of which lie in the same sheet of Σ in the neighborhood in question.

If the equation of (p, q) is given in the form

$$\phi \equiv V_p + \lambda \,.\, V_{p-1} + \ldots\ldots + \lambda^{p-\iota-1} \,.\, V_{\iota+1} + \lambda^{p-\iota} \,.\, V_\iota = 0,$$

the tangent conic at any point P' of the double line in the second sheet of Σ, that point being of the first order of multiplicity on (p, q), is a definite curve whose equation is

$\lambda \,.\, V'_{p-1} + \mu \,.\, \frac{\partial V'_p}{\partial \mu'} + \nu \,.\, \frac{\partial V'_p}{\partial \nu'} = 0$. If P' is not the pinch-point, $\frac{\partial V'_p}{\partial \nu'} \neq 0$, and the tangent conic cannot reduce to a tangent line. But if P' is the pinch-point, then must $p \geqq q+1$, and if V_p be written in the form:

$$V_p \equiv a_0\mu^p + a_1\mu^{p-1}\nu + \ldots\ldots + a_{q-1}\mu^{p-q+1}\nu^{q-1} + a_q\mu^{p-q}\nu^q,$$

it is necessary that $a_q = 0$ and $a_{q-1} \neq 0$ to insure that the curve pass once through the pinch-point in the second sheet. The pinch-point here is given by $\lambda' = 0$, $\mu' = 0$; V'_{p-1} contains the quantity μ' $p - q - 1$ times as a factor, while $\frac{\partial V'_p}{\partial \mu'}$ and $\frac{\partial V'_p}{\partial \nu'}$ contain the factors μ'^{p-q} and μ'^{p-q+1} respectively; hence the tangent curve at the pinch-point in the second sheet has for its equation $\lambda = 0$ and consists of the generator in that sheet, the double line itself; consequently any (p, q) passing once through the pinch-point in the second sheet must take the direction of the double line at that point. That such contact with the double line can occur, without having the curve meet the double line in the second sheet in two consecutive points, is evident, since the vanishing of a_q above shows that the terms containing ν^q all contain λ, and hence that the curve (p, q) has a point coincident with the pinch-point in the first sheet also; therefore a (p, q), passing once through the pinch-point in the second sheet, has contact with the double line at that point, the second point of intersection lying, not in the second, but in the first sheet there. Conversely, a (p, q) passing once through the pinch-point in the first sheet passes through the pinch-point in the second sheet also, and has the direction of the double line there, the two points of intersection with that line lying consecutively, one in one sheet and the other in the other sheet.

If the point P', determined on the double line in the second sheet of Σ by $\lambda' = 0$, $a\mu' - b\nu' = 0$, $(b \neq 0)$, is a double point of the curve (p, q), the equation of (p, q) can be written in the form

$$\phi \equiv (a\mu - b\nu)^2 . V_{p-2} + \lambda (a\mu - b\nu) . W_{p-2} + \lambda^2 . X_{p-2} + \cdots + \lambda^{p-\iota} . X_\iota = 0 .$$

That P' be a double point demands that $q \geqq 2$; hence, the tangent curve here is a tangent quartic (2, 2) whose equation is

$$\lambda^2 X'_{p-2} + \lambda\mu . a W'_{p-2} + \mu^2 . a^2 V'_{p-2} - \lambda\nu . b W'_{p-2} - 2\mu\nu . ab V'_{p-2} + \nu^2 . b^2 V'_{p-2} = 0 .$$

If $X_{p-2} \equiv (a\mu - b\nu) . Y_{p-3}$, the equation of the tangent quartic becomes

$$[\lambda . W'_{p-2} + (a\mu - b\nu) . V'_{p-2}](a\mu - b\nu) = 0,$$

and one branch of the curve at P' has the direction of the conic (1, 1) whose equation is $a\mu - b\nu = 0$, while the other branch is tangent to the conic (1, 1) given by $\lambda . W'_{p-2} + \mu . a V'_{p-2} - \nu . b V'_{p-2} = 0$, the tangent quartic breaking up into two tangent conics of the species (1, 1). If $W_{p-2} \equiv (a\mu - b\nu) . Z_{p-3}$, the equation of the tangent quartic becomes $\lambda^2 . X'_{p-2} + (a\mu - b\nu)^2 . V'_{p-2} = 0$, and the

two branches of (p, q) have at P' the directions of the two conic (1, 1)'s given by

$$\lambda . \sqrt{\frac{X'_{p-2}}{V'_{p-2}}} + a\mu - b\nu = 0 \text{ and } \lambda . \sqrt{\frac{X'_{p-2}}{V'_{p-2}}} - a\mu + b\nu = 0 .$$

And, if, again, it happen that not only $W_{p-2} = (a\mu - b\nu) . Z_{p-3}$, but also $X_{p-2} \equiv (a\mu - b\nu) . Y_{p-3}$, the equation of the tangent quartic reduces to $(a\mu - b\nu)^2 . V'_{p-2} = 0$, and the tangent curve consists of the conic (1, 1) given by $a\mu - b\nu = 0$ occurring twice. So long as $b \neq 0$, it is clear that the curve cannot have contact with the generator $\mu = 0$, and hence cannot take the direction of the double line at any point of that line apart from the pinch-point. But if $b = 0$, and P' consequently be the pinch-point in the second sheet, the equation of the tangent quartic takes the form

$$\lambda^2 . X'_{p-2} + \lambda\mu . a W'_{p-2} + \mu^2 . a^2 V'_{p-2} = 0 ,$$

in which the value zero is still to be introduced for μ'. It is clear that μ' occurs as a factor $p - q - 2$, $p - q - 1$ and $p - q$ times respectively in X'_{p-2}, W'_{p-2}, and V'_{p-2}; hence, the tangent curve consists of the double line itself in the second sheet as given twice by the equation $\lambda^2 = 0$. Here must V_p, as expressed on page 228, have $a_q = a_{q-1} = 0$, while $a_{q-2} \neq 0$; consequently, the terms of $\phi = 0$ containing ν^q all contain λ as a factor, and the curve (p, q), which passes twice through the pinch-point in the second sheet of Σ, contains the pinch-point in the first sheet at least once. Similar results are readily obtainable when P' is a point of any higher order of multiplicity on (p, q); and, in general, it is true that any branch of (p, q) which meets the double line in the second sheet at the pinch-point takes the direction of that line there, but at no other point of that line in the second sheet is the same true for any branch of (p, q).

If the equation of (p, q) be taken in the form

$$\phi \equiv U_p + \nu . U_{p-1} + \cdots + \nu^{q-1} . U_{p-q+1} + \nu^q . U_{p-q} = 0 ,$$

the direction of the curve at any point P', of the first order of multiplicity on the curve and not lying on the double line in the first sheet of Σ, is given by the tangent conic (1, 1) at the point, the equation of that conic being

$$\lambda . \frac{\partial\phi'}{\partial\lambda'} + \mu . \frac{\partial\phi'}{\partial\mu'} + \nu . \frac{\partial\phi'}{\partial\nu'} = 0 .$$

This tangent conic becomes the generator at the point when $\frac{\partial\phi'}{\partial\nu'} = 0$ and $\frac{\partial\phi'}{\partial\lambda'} \neq 0$ or $\frac{\partial\phi'}{\partial\mu'} \neq 0$.

If P' lies on the double line in the first sheet: It has been seen on page 182 that the points where (p, q) meets that line are given by $U_{p-q}=0$, and it is supposed that P' lies at the point where one of the $p-q$ generators given by that equation meets the double line, and that such an one of those points is chosen as shall be of the first order on (p, q) and shall not lie at the pinch-point. It is clear that the tangent curve to be found here is not a tangent conic (1, 1) nor a tangent quartic (2, 2), since those curves do not contain any one of the points in question; but the tangent must rather be a curve having $p \geqq q-1$. To obtain the equation of this tangent curve at the point P', the following method is available: Since $\nu/\lambda=\nu/\mu=\infty$ all along the line in question, if $1/\lambda$, $1/\mu$ and $1/\nu$ be substituted for λ, μ and ν respectively in the equation $\phi=0$, the problem becomes that of finding the tangent curve to the curve represented by the new equation thus obtained, at a point on the line given by $\nu/\lambda=\nu/\mu=0$, which point, $\overline{P'}$ is of the, first order on the curve in question, and is itself given by $\nu=0$, $\lambda/\mu=\rho$, where ρ is finite. This problem is analogous to that of finding the asymptotes of a plane curve. The new equation, obtained by the performance in $\phi=0$ of the change of variables proposed above and cleared of fractions by multiplying by $\lambda^p\mu^p\nu^q$, may be designated by $\overline{\phi}=0$ and has the form

$$\overline{\phi} \equiv \nu^q.\overline{U}_p + \nu^{q-1}.\overline{U}_{p+1} + \cdots\cdots + \nu.\overline{U}_{p+q-1} + \overline{U}_{p+q} = 0.$$

The tangent curve at P' is evidently a tangent conic (1, 1), in general, whose equation is

$$\lambda.\frac{\partial \overline{U}'_{p+q}}{\partial\lambda'} + \mu.\frac{\partial \overline{U}'_{p+q}}{\partial\mu'} + \nu.\overline{U}'_{p+q-1} = 0.$$

U_{p-q} has the form

$$U_{p-q} \equiv (a_1\lambda + b_1\mu).(a_2\lambda + b_2\mu)\cdots\cdots(a_{p-q}\lambda + b_{p-q}\mu),$$

and, accordingly,

$$\overline{U}_{p+q} \equiv \lambda^q\mu^q.(a_1\mu + b_1\lambda).(a_2\mu + b_2\lambda)\cdots\cdots(a_{p-q}\mu + b_{p-q}\lambda) \equiv \lambda^q\mu^q.\overline{\overline{U}}_{p-q},$$

where $\overline{\overline{U}}_{p-q}$ denotes what U_{p-q} becomes if λ and μ are substituted for μ and λ respectively therein. Similarly, it is seen that

$$\overline{U}_{p+q-1} \equiv \lambda^{q-1}\mu^{q-1}.\overline{\overline{U}}_{p-q+1},$$

where $\overline{\overline{U}}_{p-q+1}$ has the same relation to U_{p-q+1} as $\overline{\overline{U}}_{p-q}$ to U_{p-q}. Hence, the

equation of the tangent conic above can be expressed in the form

$$\lambda \, . \, \frac{\partial \, (\lambda'^{q} \mu'^{q} \, \overline{\overline{\overline{U'}}}_{p-q})}{\partial \lambda'} + \mu \, . \, \frac{\partial \, (\lambda'^{q} \mu'^{q} \, \overline{\overline{\overline{U'}}}_{p-q})}{\partial \mu'} + \nu \, . \, \lambda'^{q-1} \mu'^{q-1} \, \overline{\overline{\overline{U'}}}_{p-q+1} = 0 \, ,$$

or

$$\lambda \, . \, \frac{\partial \, \overline{\overline{\overline{U'}}}_{p-q}}{\partial \lambda'} + \mu \, . \, \frac{\partial \, \overline{\overline{\overline{U'}}}_{p-q}}{\partial \mu'} + \nu \, . \, \frac{\overline{\overline{\overline{U'}}}_{p-q+1}}{\lambda' \mu'} = 0 \, .$$

If, now, λ', μ', and ν' be substituted for $1/\lambda'$, $1/\mu'$, and $1/\nu'$ respectively here, and, likewise, λ, μ, and ν for $1/\lambda$, $1/\mu$, and $1/\nu$ respectively, the equation of the tangent conic (1, 1) to the curve given by $\overline{\phi} = 0$ at the point $\overline{P'}$ becomes the equation of the tangent cubic (2, 1) to the curve (p, q) at the point P', that equation being of the form

$$\lambda\mu \, . \, \frac{\overline{\overline{\overline{U'}}}_{p-q+1}}{\lambda' \mu'} + \lambda\nu \, . \, \frac{\partial \, \overline{\overline{\overline{U'}}}_{p-q}}{\partial \lambda'} + \mu\nu \, . \, \frac{\partial \, \overline{\overline{\overline{U'}}}_{p-q+1}}{\partial \mu'} = 0 \, .$$

If $\dfrac{U'_{p-q+1}}{\lambda'\mu'} = 0$ at P', the tangent cubic becomes a tangent line, the generator at P', given by $\lambda \, . \, \dfrac{\partial \, U'_{p-q}}{\partial \lambda'} + \mu \, . \, \dfrac{\partial \, U'_{p-q}}{\partial \mu'} = 0$; hence, if the group of terms U_{p-q+1} be lacking in the equation $\phi = 0$, the curve (p, q) has the direction of the generator at the point P'.

In a similar way, if P' lies at the pinch-point in the first sheet, it is found that the tangent cubic is reduced to the generator at the point given by $\lambda = 0$, i. e., the double line itself in the first sheet. As already seen, the curve (p, q) in this case passes through the pinch-point in the second sheet also, and can be regarded as having its two consecutive points on the double line lying one in either sheet at this point.

Similar results are obtained if the point P' is multiple on (p, q). And, in general, it may be said that the curve (p, q) has the direction of the double line at no point of that line save the pinch-point, but at the pinch-point can have no other direction.

6. *Plückerian Equations in the Geometry on* Σ.

The equation of the tangent conic $(1, 1)$ of any curve (p, q) at the point P' has been seen to be

$$\lambda \, . \, \frac{\partial \phi'}{\partial \lambda'} + \mu \, . \, \frac{\partial \phi'}{\partial \mu'} + \nu \, . \, \frac{\partial \phi'}{\partial \nu'} = 0 \, .$$

As a locus in λ', μ', ν', this equation represents a curve of the species $(p-1, q-1)$

or $(p-1, q)$, according as $p=q$ or $p \geqq q+1$; let the equation of this curve, i. e., the above equation regarded as an equation in λ', μ', ν', be denoted by $\psi=0$. If the number of intersections of the two curves given by the equations $\phi=0$ and $\psi=0$ be denoted by N, then may N be defined as the class of the curve (p, q), since it gives the number of tangent conics which can be drawn from any point P on Σ to the curve (p, q), each point of intersection of the two curves being the point of contact of a tangent conic from the point P to the curve (p, q). If the point P lies on (p, q), it is clear that the number in question must be diminished by two for that point. And if Δ and K represent the number of double points and of cusps, respectively, occurring on (p, q), resulting from the intersections of branches lying in the same sheet in the neighborhood in question in each case, then, as in the corresponding case of plane curves, N must be subjected to a reduction by two for each double point and by three for each cusp, giving the formula

$$N=q(2p-q-1)-2\Delta-3K$$

for the class of any curve (p, q) on Σ. It is evident that Δ and K refer to the singularities designated on pages 214–217 by H and β.

A point of the curve (p, q) at which the tangent conic $(1, 1)$ meets that curve in three consecutive points may be called an inflexion on Σ. If the number of such points be denoted by I, a formula for I in terms of p and q can be found thus:

If H be defined by the determinant of the second derivatives of the polynomial ϕ:

$$H \equiv \begin{vmatrix} \frac{\partial^2\phi}{\partial\lambda^2} & \frac{\partial^2\phi}{\partial\lambda\partial\mu} & \frac{\partial^2\phi}{\partial\lambda\partial\nu} \\ \frac{\partial^2\phi}{\partial\mu\partial\lambda} & \frac{\partial^2\phi}{\partial\mu^2} & \frac{\partial^2\phi}{\partial\mu\partial\nu} \\ \frac{\partial^2\phi}{\partial\nu\partial\lambda} & \frac{\partial^2\phi}{\partial\nu\partial\mu} & \frac{\partial^2\phi}{\partial\nu^2} \end{vmatrix},$$

and the curve whose equation is $H=0$ be called the Hessian of (p, q), then will every intersection of this Hessian with the curve (p, q) be, in general, for that curve, an inflexion on Σ, as that term has been defined. The Hessian of (p, q) is of the species $(3p-6, 3q-2)$, if $p \geqq q+2$, $(3p-6, 3q-3)$, if $p=q+1$ and $(3p-6, 3q-6)$, if $p=q$, as the determinant above shows. Consequently the number of intersections of the curve (p, q) with its Hessian is found to be $2p(3q-1)-q(3q+4)$, when $p \geqq q+2$, and $3p(2q-1)-3q(q+1)$, when

$p \leqq q+1$. But these numbers are subject, as in the analogous case in plane curves, to a reduction by $6\Delta + 8K$, where Δ and K have the meanings assigned them above. And, furthermore, while the formula for the intersections used above makes the necessary reductions, in general, for the double line of Σ, it is known that a plane curve and its Hessian have contact between both branches of the two curves at a double point; accordingly, whenever $p \geqq q+2$, the curve (p, q) and its Hessian have $p-q$ actual intersections on the double line in the first sheet of Σ; but these intersections are no more to be regarded as inflexions on Σ for the curve (p, q) than are the points of intersection of a plane curve with the infinite line, in general, to be regarded as inflexions in determining the number ι for the curve in the plane; therefore, a further reduction by $p-q$ is necessary when $p \geqq q+2$. Thus is obtained the formula

$$I = 3p(2q-1) - 3q(q+1) - 6\Delta - 8K,$$

which holds for all values of p and q.

The same results are obtained at once from the Plückerian formulas for the class and the number of inflections of a plane curve if the curve is supposed to have a $(p-q)$-tuple point, and n, ι, δ, $\varkappa$, and m are replaced by N, I, Δ, K, and p respectively in the formulas for n and ι; thus

$n = m(m-1) - 2\delta - 3\varkappa$ gives

$$N = p(p-1) - (p-q)(p-q-1) - 2\Delta - 3K$$

$= q(2p-q-1) - 2\Delta - 3K$; and, similarly,

$\iota = 3m(m-2) - 6\delta - 8\varkappa$ becomes

$$I = 3p(p-2) - 3(p-q)(p-q-1) - 6\Delta - 8K$$
$$= 3p(2q-1) - 3q(q+1) - 6\Delta - 8K.$$

WILLIAMS COLLEGE, *February* 1, 1900.

Congruent Reductions of Bilinear Forms.

By T. J. I'A. Bromwich.

The following paper contains an account and a slight extension of a method due to Kronecker,* which, in the first place, was employed for the reduction of two quadratic forms; this method seems to have been used by no other writer, although in some ways it is the simplest that has been proposed. Here I have applied the method to four cases of reductions: (i) two symmetric forms (the same as Kronecker's case of two quadratics); (ii) a symmetric and an alternate form; (iii) two alternate forms, and (iv) two Hermite's forms. In cases (i)–(iii) the substitutions are congruent, while in (iv) they are conjugate imaginaries.

In §§1, 2, I have explained a method† for the reduction of a single form (alternate or symmetric); in §3, I have explained Kronecker's procedure for reducing two quadratic forms, using the results obtained in §§1, 2; the method is here put into such a shape that it can be applied to cases (ii), (iii) as well as (i). This way of explaining the reduction of two quadratics is suggested by Kronecker in an addition ("Nachtrag," Ges. Werke, Bd. 1, p. 397) to the first paper quoted. I have added a supplementary method to fill up a gap which presents itself in applying Kronecker's method to case (ii). In §4 is given a list of the reduced forms obtained for cases (i)–(iii). In §5, I have considered case (iv) somewhat briefly, as it can be obtained from the other three cases.

It may be convenient to indicate here the principal papers dealing with the problems to be considered.

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* Berliner Monatsberichte, 1874, p. 59 = Ges. Werke, Bd. 1, p. 349.

† This method for symmetric forms is the same as Kronecker's in the paper just quoted; the method for alternate forms is essentially the same as Kronecker's in a second paper (Berichte, p. 397 = Ges. Werke, p. 421).

Case (i). Weierstrass, Berliner Monatsberichte, 1858, p. 207, and 1868, p. 310 = Ges. Werke, Bd. 1, p. 233 and Bd. 2, p. 19.

Kronecker, Berliner Monatsberichte, 1868, p. 339 = Ges. Werke, Bd. 1. p. 163.

Berliner Sitzungsberichte, 1890, pp. 1225, 1375; 1891, pp. 9, 33, and the first papers quoted above.

Darboux, Liouville's Journal, t. 19 (sér. 2), 1874, p. 347.

Jordan, ibid., p. 397.

Case (ii). Kronecker, in the second paper* quoted above (i. e., Ges. Werke, Bd. 1, p. 421).

Frobenius, Berliner Sitzungsberichte, 1896, p. 7.

Case (iii). Frobenius, Crelle's Journal, Bd. 86, 1879, p. 140 (§§7, 13), and in the paper last quoted.

E. v. Weber, Münchener Sitzungsberichte, 1898, p. 369.

Case (iv). Alf. Loewy, Crelle's Journal, Bd. 122, 1900, p. 53.

Frobenius's paper (Berl. Ber., 1896, p. 7) contains a general theorem that if any two substitutions are known to change two forms A, B into two others C, D, then a congruent substitution can be deduced, which will make the same transformation, provided that A, C are both symmetric or both alternate, and that B, D have the same property. Thus (by virtue of Weierstrass's general theory), if A, B are given, they can be transformed into C, D by a *congruent* substitution, provided that the invariant-factors of $|\lambda A - B|$, $|\lambda C - D|$ are the same; this condition is obviously *necessary*, but Frobenius proves that it is also *sufficient*. In a paper recently published,† I have shown how to modify Weierstrass's process so as to obtain the congruent substitutions directly, thus giving an independent verification of Frobenius's results.

Apart from the immediate algebraical interest of these problems, they have certain applications, some of which I have indicated elsewhere,‡ and I hope that the solution given here may not be found superfluous.

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* Kronecker's object in this paper was to reduce a single bilinear form by congruent substitutions; not a symmetric and an alternate form simultaneously. The two problems are, however, essentially the same. The origin of the problem was connected with Weierstrass's generalized theta-functions (cf. Kronecker, Berliner Monatsberichte, 1866, p. 597 = Crelle, Bd. 68. p. 273 = Werke. Bd. 1, p. 143).

† Proc. Lond. Math. Soc., vol. XXXII, 1900, p. 321. This paper also contains a solution of Case (iv) and some applications to automorphic substitutions of bilinear forms.

‡ See last reference; two papers on dynamical applications will appear shortly in the same Proceedings.

1. Let $A = \Sigma a_{rs} x_r y_s$ be a bilinear function of the $2n$ variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ and now examine the form

$$B = \begin{vmatrix} a_{11}, & \ldots, & a_{1k}, & \dfrac{\partial A}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk}, & \dfrac{\partial A}{\partial x_k} \\ \dfrac{\partial A}{\partial y_1}, & \ldots, & \dfrac{\partial A}{\partial y_k}, & A \end{vmatrix}.$$

Differentiating with respect to x_r, we see that (since x_r appears only in the last row of B),

$$\frac{\partial B}{\partial x_r} = \begin{vmatrix} a_{11}, & \ldots, & a_{1k}, & \dfrac{\partial A}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk}, & \dfrac{\partial A}{\partial x_k} \\ a_{r1}, & \ldots, & a_{rk}, & \dfrac{\partial A}{\partial x_r} \end{vmatrix},$$

which vanishes identically if $r = 1, 2, \ldots, k$. Thus B does not contain $x_1, x_2, \ldots, x_k$; and by a similar method B does not contain $y_1, y_2, \ldots, y_k$. In particular, if $k = n$, we have $B = 0$, and we obtain a familiar form for A.

Again, by examining $\dfrac{\partial^2 B}{\partial x_r \partial y_s}$, it is easy to prove that B vanishes identically if *all* the $(k+1)$-rowed determinants of the type

$$\begin{vmatrix} a_{11}, & \ldots, & a_{1k}, & a_{1s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk}, & a_{ks} \\ a_{r1}, & \ldots, & a_{rk}, & a_{rs} \end{vmatrix} \qquad (r, s > k)$$

are zero; and, in this case, if the k-rowed determinant

$$\begin{vmatrix} a_{11}, & \ldots, & a_{1k} \\ \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk} \end{vmatrix}$$

is not zero, A is expressible in terms of the $2k$ quantities

$$\frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_k}, \frac{\partial A}{\partial y_1}, \ldots, \frac{\partial A}{\partial y_k}.$$

In particular, if the determinant $|A|$ of the form A is zero, it is possible to express A in terms of $2(n-1)$ variables, or fewer variables, in case *all* the first minors are zero too.

We shall have occasion to apply the above theorem in various forms. Take in the first place $k=1$, and then

$$A = \frac{1}{a_{11}} \frac{\partial A}{\partial y_1} \frac{\partial A}{\partial x_1} + \frac{1}{a_{11}^2} B,$$

where B does not contain x_1 or y_1.

Supposing that the variables are not subject to the condition of undergoing congruent substitutions, we can always apply this method, provided that x_1 does appear in A, for then x_1 multiplies at least one y, and we may call this y, y_1, and so $a_{11} \neq 0$, which is the condition for the application of our result. Pass then to the case of congruent substitutions; here A will be either symmetrical or alternate, and we proceed to examine these cases.

First, take a symmetrical form, then, if $a_{11} \neq 0$, we can write

$$A = \frac{1}{a_{11}} \frac{\partial A}{\partial y_1} \frac{\partial A}{\partial x_1} + \frac{1}{a_{11}^2} B,$$

where B is again symmetrical and does not contain x_1 or y_1.

But if $a_{11}=0$, we must proceed to the result given by $k=2$, and then we find (since $a_{12}=a_{21}$),

$$A = \frac{1}{a_{12}} \left(\frac{\partial A}{\partial y_1} \frac{\partial A}{\partial x_2} + \frac{\partial A}{\partial y_2} \frac{\partial A}{\partial x_1} \right) - \frac{a_{22}}{a_{12}^2} \frac{\partial A}{\partial y_1} \frac{\partial A}{\partial x_1} - \frac{1}{a_{12}^2} B.$$

Thus, if we write

$$X_2 = \frac{\partial A}{\partial y_1}, \qquad X_1 = \frac{1}{a_{12}} \frac{\partial A}{\partial y_2} - \tfrac{1}{2} \frac{a_{22}}{a_{12}^2} \frac{\partial A}{\partial y_1},$$

$$Y_2 = \frac{\partial A}{\partial x_1}, \qquad Y_1 = \frac{1}{a_{12}} \frac{\partial A}{\partial x_2} - \tfrac{1}{2} \frac{a_{22}}{a_{12}^2} \frac{\partial A}{\partial x_1},$$

we have

$$A = X_1 Y_2 + X_2 Y_1 - \frac{1}{a_{12}^2} B,$$

where B is a symmetrical form, not containing x_1, y_1, x_2, y_2. The condition that this method should be applicable is that $a_{12} \neq 0$, and when x_1, y_1 have been fixed upon, it is always possible to find x_2, y_2, so that $a_{12} \neq 0$, provided that x_1, y_1 do

actually appear in A. It will be observed that X_1, Y_1 may contain all the variables, while X_2, Y_2 do not contain x_1, y_1.

In the alternate case $a_{11}=0$, $a_{22}=0$, and $a_{12}=-a_{21}$; thus taking $k=2$, we find

$$A=\frac{1}{a_{12}}\left(\frac{\partial A}{\partial x_1}\frac{\partial A}{\partial y_2}-\frac{\partial A}{\partial x_2}\frac{\partial A}{\partial y_1}\right)+\frac{1}{a_{12}^2}B.$$

Now, if we write

$$X_1=\frac{1}{a_{12}}\frac{\partial A}{\partial y_2},\qquad Y_1=-\frac{1}{a_{12}}\frac{\partial A}{\partial x_2},$$

$$X_2=\frac{\partial A}{\partial y_1},\qquad Y_2=-\frac{\partial A}{\partial x_1},$$

the substitutions are congruent and the result is

$$A=-X_1Y_2+X_2Y_1+\frac{1}{a_{12}^2}B,$$

where B is an alternate form not containing x_1, y_1, x_2, y_2; here X_1 contains x_1 and may contain all the other x's except x_2, while X_2 contains x_2 and may contain all but x_1.

Suppose, now, that the variables are divided in any way into two sets G, H; owing to the congruent conditions, it will be necessary to suppose that y_r belongs to the same set as x_r. Let us now impose the condition that the variables in H can only be linearly combined amongst themselves, and that no variables from G may be added to the variables in H; on the other hand, variables from H may be added freely to those in G. Then, applying the methods just explained,* we get a series of reduced terms (in which each variable occurs once only) of the type $(x_ry_s+x_sy_r)$ or $(x_ry_s-x_sy_r)$, according as the form considered is symmetric or alternate; in the process, the sets of variables G, H will be further subdivided into G_1, G_2, H_1, H_2. The characteristic of G_1 is that its variables multiply each other only, while those of G_2 multiply those of H_1; the variables in H_2 multiply each other. We may write the reduced form symbolically

$$A=(G_1)+(G_2H_1)+(H_2).$$

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* The pair of variables represented above by x_1, y_1 should be taken always from the variables in G, then X_1, Y_1 will replace x_1, y_1 in G; and if we have the reduced form $(X_1Y_2+X_2Y_1)$ or $(X_1Y_2-X_2Y_1)$, we have to examine X_2, Y_2, then if any variables from G appear in X_2, Y_2, the quantities X_2, Y_2 will be taken as new variables, replacing two of those in G. But it may happen that X_2, Y_2 contain only variables from H (as they do *not* contain x_1, y_1), and then they form part of H_1; in this case X_1, Y_1 are variables belonging to the substitution G_2.

2. Again, returning to our original theorem, let us take $k = n - 1$, and

$$B = \begin{vmatrix} a_{22} , & \ldots , & a_{2n} , & \dfrac{\partial A}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2} , & \ldots , & a_{nn} , & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_2} , & \ldots , & \dfrac{\partial A}{\partial y_n} , & A \end{vmatrix} ,$$

then B will contain only x_1 and y_1. To evaluate B, differentiate with respect to x_1 and we see that

$$\frac{\partial B}{\partial x_1} = y_1 D, \text{ or } B = D x_1 y_1 ,$$

where D is the determinant $|a_{rs}|$, $(r, s = 1, 2, \ldots, n)$. Thus we have

$$A D_{11} = D x_1 y_1 - \begin{vmatrix} a_{22} , & \ldots , & a_{2n} , & \dfrac{\partial A}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2} , & \ldots , & a_{nn} , & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_2} , & \ldots , & \dfrac{\partial A}{\partial y_n} , & 0 \end{vmatrix} ,$$

where D_{11} is the minor of a_{11} in D.

But, if D_{11} is not zero, it is possible to find $X_2, \ldots, X_n$ so as to satisfy

$$a_{22} X_2 + \ldots + a_{n2} X_n = \frac{\partial A}{\partial y_2} ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{2n} X_2 + \ldots + a_{nn} X_n = \frac{\partial A}{\partial y_n} ,$$

and then each of the $(n - 1)$ differences $(X_2 - x_2), \ldots, (X_n - x_n)$ is a multiple of x_1. Similarly, we can find $(n - 1)$ y's to satisfy the equations

$$a_{22} Y_2 + \ldots + a_{2n} Y_n = \frac{\partial A}{\partial x_2} ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n2} Y_2 + \ldots + a_{nn} Y_n = \frac{\partial A}{\partial x_n} .$$

Then we can clearly write

$$A = \frac{D}{D_{11}} x_1 y_1 + C,$$

where C is derived from A by writing zero for x_1, y_1 and X_r, Y_r for x_r, y_r $(r = 2, 3, \ldots, n)$. We see that the equations for X_r, Y_r may be put in the form $\frac{\partial C}{\partial Y_r} = \frac{\partial A}{\partial y_r}, \frac{\partial C}{\partial X_r} = \frac{\partial A}{\partial x_r}$. This method can, of course, be applied if $D = 0$, but cannot be applied if $D_{11} = 0$. As explained before, if there is no restriction as to congruent substitutions, we shall, in general, be able to arrange the y's corresponding to a given x, so that $D_{11} \neq 0$; an exceptional case may arise if *all* the first minors of D are zero; but then it is possible to reduce A so as to depend only on $(n - 2)$ pairs of variables (or fewer pairs).

But, in symmetrical or alternate cases, it cannot always be arranged, that $D_{11} \neq 0$, and we proceed to examine these cases. Suppose that A is symmetrical, then if $D_{11} \neq 0$, our result still holds and C is symmetrical; but if $D_{11} = 0$, while $D \neq 0$, consider the form (containing only x_1, y_1, x_2, y_2),

$$B = \begin{vmatrix} a_{33}, & \ldots, & a_{3n}, & \frac{\partial A}{\partial x_3} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n3}, & \ldots, & a_{nn}, & \frac{\partial A}{\partial x_n} \\ \frac{\partial A}{\partial y_3}, & \ldots, & \frac{\partial A}{\partial y_n}, & A \end{vmatrix},$$

We find on calculation that

$$\frac{\partial B}{\partial x_1} = y_1 D_{22} - y_2 D_{21}, \quad \frac{\partial B}{\partial x_2} = -y_1 D_{12} + y_2 D_{11},$$

where D_{12}, D_{21}, D_{22} are the minors of a_{12}, a_{21}, a_{22} respectively in D. But $D_{11} = 0$ and $D_{12} = D_{21}$ from symmetry and so

$$B = x_1 y_1 D_{22} - D_{12}(x_1 y_2 + x_2 y_1),$$

for B contains none of the variables $x_3, \ldots, x_n, y_3, \ldots, y_n$. When B is expanded in terms of A and the products $\frac{\partial A}{\partial x_r} \frac{\partial A}{\partial y_s}$, the coefficient of A is a second principal minor of D; and the complementary minor of D is

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}.$$

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Hence the minor multiplying A in the expression for B is equal to

$$\frac{1}{D}\begin{vmatrix} D_{11}, & D_{12} \\ D_{21}, & D_{22} \end{vmatrix} = -D_{12}^2/D; \text{(since } D_{11}=0).$$

Thus

$$A = \frac{D}{D_{12}}(x_1y_2 + x_2y_1) - \frac{DD_{22}}{D_{12}^2}x_1y_1 + \frac{D}{D_{12}^2}\begin{vmatrix} a_{33}, & \dots, & a_{3n}, & \dfrac{\partial A}{\partial x_3} \\ \dots & \dots & \dots & \dots \\ a_{n3}, & \dots, & a_{nn}, & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_3}, & \dots, & \dfrac{\partial A}{\partial y_n}, & 0 \end{vmatrix},$$

or, if we write $X_2 = x_2 - \frac{1}{2}\frac{D_{22}}{D_{12}}x_1$, $Y_2 = y_2 - \frac{1}{2}\frac{D_{22}}{D_{12}}y_1$, the first terms in A can can be put in the shape

$$\frac{D}{D_{12}}(x_1Y_2 + y_1X_2).$$

Now, if $D_{12} \neq 0$, the second minor of D,

$$\begin{vmatrix} a_{33}, & \dots, & a_{3n} \\ \dots & \dots & \dots \\ a_{n3}, & \dots, & a_{nn} \end{vmatrix}$$

will not vanish, for this has been proved to be $-D_{12}^2/D$; and so the $(n-2)$ quantities $X_3, \dots, X_n$ can be found to satisfy the $(n-2)$ equations

$$a_{33}X_3 + \dots + a_{n3}X_n = \frac{\partial A}{\partial y_3},$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{3n}X_3 + \dots + a_{nn}X_n = \frac{\partial A}{\partial y_n},$$

and each of the differences $(X_3 - x_3), \dots, (X_n - x_n)$ will be a linear function of x_1, x_2. By symmetry, we have similar equations to determine $Y_3, \dots, Y_n$, and then, on substitution in the expression for A, we find

$$A = \frac{D}{D_{12}}(x_1Y_2 + y_1X_2) + C,$$

where C is found from A by writing zero for x_1, y_1, x_2, y_2 and X_r, Y_r in place of x_r, y_r $(r = 3, 4, \dots, n)$. Here again X_r, Y_r can be determined by the equations $\dfrac{\partial C}{\partial Y_r} = \dfrac{\partial A}{\partial y_r}$, $\dfrac{\partial C}{\partial X_r} = \dfrac{\partial A}{\partial x_r}$.

It will, of course, be possible always to find one minor D_{12} which does not vanish unless D is itself zero. But if D be zero, we shall be able to reduce A to depend only on $2(n-1)$ variables instead of $2n$.

Now, consider the alternate case; here if n be odd, the determinant D is zero; and if n be even, $D_{11}, D_{22}, \ldots,$ which are skew-symmetrical determinants of odd order, vanish.

If n be odd, we can apply our first method without any special modification to show that A can be brought to a form containing only $2(n-1)$ variables. Thus we consider only the case when n is even. Suppose that $D \neq 0$ (or we could reduce the number of variables to $2(n-2)$), and then $D_{11}=0$, $D_{22}=0$, $D_{12}=-D_{21}$. So, for the form

$$B = \begin{vmatrix} a_{33}, & \ldots, & a_{3n}, & \dfrac{\partial A}{\partial x_3} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n3}, & \ldots, & a_{nn}, & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_3}, & \ldots, & \dfrac{\partial A}{\partial y_n}, & A \end{vmatrix},$$

we find the value $B=(x_1y_2-x_2y_1)D_{12}$. Just as before, we can find $X_3, \ldots, X_n$ to satisfy the $(n-2)$ equations

$$a_{33}X_3 + \ldots + a_{n3}X_n = \frac{\partial A}{\partial y_3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{3n}X_3 + \ldots + a_{nn}X_n = \frac{\partial A}{\partial y_n},$$

and, owing to the alternate property $(a_{rs}=-a_{sr})$, it follows that $Y_3, \ldots, Y_n$ will satisfy

$$a_{33}Y_3 + \ldots + a_{3n}Y_n = \frac{\partial A}{\partial x_3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n3}Y_3 + \ldots + a_{nn}Y_n = \frac{\partial A}{\partial x_n}.$$

Then we have, on substitution,

$$A = \frac{D}{D_{12}}(x_1y_2-x_2y_1)+C,$$

where C is found by writing zero for x_1, y_1, x_2, y_2 in A, and X_r, Y_r for x_r, y_r $(r = 3, 4, \ldots, n)$.

We can express this result in a slightly different form by using Pfaffians; we know that D can be put in the shape

$$D = P^2,$$

where P is a rational function of the elements of the determinant; and the second principal minor, which is equal to $+D_{12}^2/D$, is also the square of a Pfaffian, say of P_1. Hence, $D_{12} = \pm PP_1$, and we can determine the sign of P_1 so as to give the upper sign, thus we shall have

$$A = \frac{P}{P_1}(x_1 y_2 - x_2 y_1) + C.$$

Let the variables be divided in any way into two sets H, K (as before, x_r, y_r belong to the same set); here suppose that the variables in H may only be combined with each other, while those in K may be combined with each other and with those in H as well. We apply the process given above: The sets then each subdivide into two; H into H_1, H_2; and K into K_1, K_2; the variables in H_1 (in the reduced forms) multiply other variables of the set H_1; those in H_2 and K_1 are multiplied together.* Thus using the same symbolical notation as before, we have

$$A = (H_1) + (H_2 K_1) + (K_2).$$

3.—*General account of Kronecker's Method of Reduction.*

Consider two forms A, B which may be either symmetric or alternate, so that each of them can be brought to a reduced form by means of congruent substitutions of the types already indicated. We shall suppose that A contains some variables that do not appear in B, while B also contains some that are not present in A. This divides the variables into three sets, G, H, K; G contains all the variables that appear in A and do not appear in B; K, those that are in B and not in A; H, those that are common to the two forms.

Applying the first method of reduction (§1) to A, the variables G are divided into G_1, G_2, and the variables H into $H_1 = L$ and $H_2 = M$. Of course,

* In the foregoing reduction, we select x_1, y_1 from the variables in H; if the reduced part is $(x_1 Y_2 \pm X_2 y_1)$, we examine X_2, Y_2; in case X_2, Y_2 contain only variables from H, we group x_1, y_1, X_2, Y_2 in H_1, but if X_2, Y_2 contain variables from K, x_1, y_1 will belong to H_2, and X_2, Y_2 to K_1.

in the reduction, the variables are modified, but in such a way that the new variables in H are linear functions only of the old variables in H. Then A takes the form $(G_1)+(G_2L)+(M)$, where, in the forms (G_1), (G_2L), each variable occurs once only.

Substitute these new variables in the form B and consider then the variables of B as belonging to the three sets L, M, K. We now apply the second method of reduction (§2) to B and we shall obtain a result of the form

$$(L_1)+(L_2K_1)+(L_3M_1)+(M_2)+(K_2),$$

the variables L dividing into L_1, L_2, L_3, M into M_1, M_2, and K into K_1, K_2. In this process the L's are modified only by other L's, and the M_1's only by M_1's and L's; further, in the terms (L_1), (L_2K_1), (L_3M_1), each variable occurs once only. Substitute the new variables in the expression for A; the additional terms introduced by the change of variables will either contain only M's or else will have some variable L as a factor; in the latter case the additional terms can be absorbed by modifying the variables G_2. Thus we may now write

$$\begin{aligned} A &= (G_1)+(G_2L)+(M) \\ B &= \qquad\qquad (L_1)+(L_2K_1)+(L_3M_1)+(M_2)+(K_2) \end{aligned}$$

in these we can pair off certain terms. Thus we remove

$$(G_1),\ (G_2L_1) \text{ and } (L_1),\ (G_2L_2) \text{ and } (L_2K_1),$$

and the remainders will take the forms

$$\begin{aligned} &(G_2L_3)+(M) \\ &(L_3M_1)+(M_2)+(K_2). \end{aligned}$$

Now, taking the terms (M) and $(M_2)+(K_2)$, we observe that they are of the same general type as A and B were at first, but contain fewer variables; and the groups of variables corresponding to the original G, H, K are here M_1, M_2, K_2. We can accordingly continue the process as given above, and in the continuation we alter none of the variables in those parts of A, B, which have been already reduced, except those which belong to the set M; those in M_1 may have to be replaced by linear functions of the M's, when we reduce the two (M) and $(M_2)+(K_2)$.

So far as the variables M_1 are altered by substitutions containing only themselves, we can make corresponding substitutions on the variables L_3 and G_2, so that

$$(G_2'L_3')=(G_2L_3) \text{ and } (L_3'M_1')=(L_3M_1).$$

Thus we have only to examine the effect of adding on variables from M_2 to the variables M_1; this will give, instead of

$$(L_3M_1) + (M_2) + (K_2),$$

the terms
$$(L_3M_1) + (L_3M_2) + (M_2) + (K_2).$$

Now, by using the second method of reduction, we can combine $(L_3M_2)+(M_2)$ so that, by adding on linear functions of the variables L_3 to the variables M_2, we get $(M_2') + (L_3)$, and then these can be combined with (L_3M_1) so that we may write

$$(L_3M_1) + (L_3M_2) + (M_2) = (L_3M_1') + (M_2'),$$

where now each M is modified by some of the variables L_3. Substituting in the remaining terms of A, we find that the additional terms so introduced have each one variable L_3 as a factor and so can be combined with the terms (G_2L_3) by introducing new variables G_2.

It follows that this method can be continued so long as there are variables in one form which do not appear in the other, and that when we have to stop. there can only be variables common to both forms in the parts which remain.

If the forms to be reduced are *both* symmetric or *both* alternate, the process of reduction, as just explained, can be applied to complete the whole reduction, For, if A, B are both symmetric (or alternate), then all forms of the family $(uA + vB)$ are symmetric (or alternate), and we can determine values of u, v for which the determinant $|uA + vB|$ vanishes; say $u_1 : v_1$, $u_2 : v_2$ are two values of the ratio $u : v$ for which the determinant is zero. Then $u_1A + v_1B$, $u_2A + v_2B$ will be two forms, each a function of fewer variables than appear in the general form $uA + vB$. Thus $u_1A + v_1B$, $u_2A + v_2B$ can be transformed so that each contains some variables that do not occur in the other, and the process already sketched can be completely carried out.

But if one of the forms (A, say) be symmetric while the other (B) is alternate, then the general form $uA + vB$ will be neither symmetric nor alternate, and thus the method of reduction described can be carried out only so long as $|A| = 0$ or $|B| = 0$. We shall now explain a process of reduction to cover the general case.*

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* Kronecker describes his reduction of a single bilinear form by congruent substitutions as suggested by his method for two quadratics; but it differs considerably from that given here, although the fundamental ideas are the same.

By what has been proved, we can reduce a symmetric and an alternate form to the shapes

$$A = A_1 + A_2, \quad B = B_1 + B_2,$$

where A_1, B_1 are reduced forms and A_2, B_2, are forms of such a character that $|A_2| \neq 0$, $|B_2| \neq 0$; further A_2, B_2 will not contain the variables in A_1, B_1. Consequently, the number of pairs of variables in A_2, B_2 must be *even*.

Suppose then that we start from two forms A, B in $2m$ pairs of variables neither of whose determinants is zero; and let $\lambda = c$ be a root of the determinantal equation $|\lambda A + B| = 0$. Then, since $(cA + B)$ is a form with zero determinant, it depends on $(2m - 1)$ y's at most; and so we can choose our variables in such a way that one x (x_1 say) has no corresponding y present in $(cA + B)$, the substitutions being supposed congruent. Apply now to $(cA + B)$ the method of transformation explained in §1, for a single bilinear form; then

$$cA + B = 2cx_1'y_2' + \text{terms without } x_1',\ y_1' \text{ or } y_2',$$

where (since there is no term x_1y_1 in $cA + B$), x_1' is a linear function of x's (containing x_1) and y_2' is a linear function of y's (not containing y_1). Interchange the x's and y's of the last equation, then, as this changes the sign of B,

$$cA - B = 2c\, x_2' y_1' + \text{terms without } x_1', x_2' \text{ or } y_1'.$$

Hence
$$A = x_1' y_2' + x_2' y_1' + \text{terms without } x_1', y_1' \text{ or the product } x_2' y_2',$$
$$B = c\,(x_1' y_2' - x_2' y_1') + \text{terms without } x_1', y_1'.$$

Applying to A the method given in §1 for a single symmetrical form we can collect all the terms in x_2', y_2' together and write

$$A = x_1'' y_2' + x_2' y_1'' + \text{terms without } x_1'', y_1'', x_2', y_2',$$
$$B = c\,(x_1'' y_2' - x_2' y_1'') + \text{terms without } x_1'', y_1''.$$

Take next the pair of forms in $4(m - 1)$ variables obtained from A, B by putting $x_2' = 0$, $y_2' = 0$; we treat them in the same way. Proceeding thus we may write finally, dropping the accents,

$$A = (x_1 y_2 + x_2 y_1) + (x_3 y_4 + x_4 y_3) + \ldots\ldots + (x_{2m-1} y_{2m} + x_{2m} y_{2m-1}),$$
$$B - B_1 = c_1 (x_1 y_2 - x_2 y_1) + c_2 (x_3 y_4 - x_4 y_3) + \ldots\ldots + c_m (x_{2m-1} y_{2m} - x_{2m} y_{2m-1}),$$

where
$$B_1 = (x_2 \eta_2 - y_2 \xi_2) + \ldots\ldots (x_{2m-2} \eta_{2m-2} - y_{2m-2} \xi_{2m-2}),$$

and ξ_r is a linear function of $x_{r+1}, x_{r+2}, \ldots\ldots, x_{2m}$,
η_r being the same function of $y_{r+1}, y_{r+2}, \ldots\ldots, y_{2m}$.

We now proceed to remove as many terms as possible from B_1.

Take the term $\alpha(x_2 y_3 - x_3 y_2)$,

and write

$$x_1 = x_1' - \mu x_3, \quad y_1 = y_1' - \mu y_3,$$
$$x_4 = x_4' + \mu x_2, \quad y_4 = y_4' + \mu y_2.$$

Then

$$x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 = x_1' y_2 + x_2 y_1' + x_3 y_4' + x_4' y_3,$$

and

$$\begin{aligned} c_1(x_1 y_2 - x_2 y_1) + \alpha(x_2 y_3 - x_3 y_2) + c_2(x_3 y_4 - x_4 y_3) \\ = c_1(x_1' y_2 - x_2 y_1') + c_2(x_3 y_4' - x_4' y_3) + (\alpha + \mu c_1 - \mu c_2)(x_2 y_3 - x_3 y_2), \end{aligned}$$

Or, if $\mu = \alpha / (c_2 - c_1)$, the term in question will be removed from B by the substitution proposed; but if $c_1 = c_2$, this will be no longer possible. We may accordingly regard A and B as reduced, except for terms of the type

$$(A) \qquad (x_1 y_2 + x_2 y_1) + \ldots\ldots + (x_{2p-1} y_{2p} + x_{2p} y_{2p-1})$$
$$(B) \qquad c(x_1 y_2 - x_2 y_1) + \ldots\ldots + c(x_{2p-1} y_{2p} - x_{2p} y_{2p-1}) + B_2.$$

where B_2 contains terms of the same type as B_1 before, but is limited to the variables $x_2, \ldots\ldots, x_{2p}, y_2, \ldots\ldots, y_{2p}$. Then consider a term in B_2 of the type

$$\alpha(x_2 y_{2q} - x_{2q} y_2),$$

and write

$$x_1 = x_1' + (\alpha / 2c) x_{2q},$$
$$x_{2q-1} = x_{2q-1}' - (\alpha / 2c) x_2,$$

with the corresponding substitution in the y's. Then the form of A is unaltered and the terms affected in B become

$$\begin{aligned} cy_2[x_1' + (\alpha/2c)x_{2q}] - cx_2[y_1' + (\alpha/2c)y_{2q}] + cy_{2q}[x_{2q-1}' - (\alpha/2c)x_2] \\ - cx_{2q}[y_{2q-1}' - (\alpha/2c)y_2] + \alpha(x_2 y_{2q} - x_{2q} y_2) \\ = c(x_1' y_2 - x_2 y_1') + c(x_{2q-1}' y_{2q} - x_{2q} y_{2q-1}'). \end{aligned}$$

By the same method we remove all terms with two even suffixes from B_2; so proceeding in this way, the only terms left in B_2 will be of the type

$$x_{2r} y_{2q+1} - y_{2r} x_{2q+1}, \qquad (q \geqq r).$$

Consider those in x_2, y_2; let us suppose that the first term of B_2 which contains x_2, y_2 is

$$x_2 y_3 - x_3 y_2;$$

this assumption may involve certain changes in the suffixes; but it will not

interchange the even and odd suffixes. It may also require a division and multiplication of certain variables by a constant. Consider next the pair of terms in B_2.

$$x_2 y_3 - x_3 y_2 + \alpha (x_2 y_{2q-1} - y_2 x_{2q-1}), \qquad (q > 2).$$

Write
$$x_3' = x_3 + \alpha x_{2q-1},\ x_{2q}' = x_{2q} - \alpha x_4,$$

then the form of A remains the same and these terms in B_2 become,

$$x_2 y_3' - x_3' y_2 .$$

This substitution may add to the parts of B_2 which contain x_4, y_4 but will not otherwise alter B. Of course it may happen that B_2 contains no terms in x_2, y_2 and then we shall not have to reduce the terms $(x_1 y_2 + x_2 y_1)$, $c(x_1 y_2 - x_2 y_1)$ in B any further.

Continuing our process, we have finally A, B divided into groups of terms

$$\begin{array}{ll} (A) & (x_1 y_2 + x_2 y_1) + \ldots\ldots + (x_{2s-1} y_{2s} + x_{2s} y_{2s-1}), \\ (B) & c(x_1 y_2 - x_2 y_1) + \ldots\ldots + c(x_{2s-1} y_{2s} - x_{2s} y_{2s-1}), \\ & + (x_2 y_3 - x_3 y_2) + \ldots\ldots + (x_{2s-2} y_{2s-1} - x_{2s-1} y_{2s-2}). \end{array}$$

In particular for $s = 1$ we have

$$\begin{array}{ll} (A) & (x_1 y_2 + x_2 y_1) \\ (B) & c(x_1 y_2 - x_2 y_1). \end{array}$$

It is readily seen that these terms correspond to invariant-factors $(\lambda - c)^s$ $(\lambda + c)^s$, of $|\lambda A - B|$.

4.—*Lists of Reduced Forms.*

We now proceed to enumerate all the possible types of reduced forms at which we arrive by following Kronecker's process as just explained. There are three cases, each of which has to be examined separately; i. e., two symmetric forms one symmetric and one alternate form; two alternate forms.

Corresponding to each type we give the invariant-factors of $|uA + vB|$, showing that the reduced types depend entirely on the invariant-factors, except in the "singular" case when $|uA + vB| \equiv 0$. The determination of the necessary invariants for the singular case is not considered here, as it has been fully explained by Kronecker and others.*

* Nearly all the papers of Kronecker's which have been quoted give some details as to the proper invariants; the 1890 and 1891 papers are the most complete. The last section of Darboux's paper may also be consulted; and a somewhat different determination of the invariants has been effected by the present writer. (Proc. Lond. Math. Soc., vol. XXXII, 1900, pp. 88 and 326.)

Two symmetric forms.

This case (which is equivalent to that of two quadratic forms) has been handled so often* that what is given here is only included to give completeness to the investigation.

The simple sets of terms are of the types

(i). (G_1): $\qquad A,\ x_1 y_1$ or $x_1 y_2 + x_2 y_1$

corresponding to an invariant-factor u, or two u, u of $|uA + vB|$. There will be no terms in B which contain x_1, y_1, x_2, y_2 in this case. In the second case the form can be reduced to

$$\tfrac{1}{2}(X_1 Y_1 - X_2 Y_2)$$

by writing $\qquad X_1 = x_1 + x_2,\ X_2 = x_1 - x_2.$

(ii). $(G_2 L_1)$ and (L_1): $\qquad A,\ x_1 y_2 + x_2 y_1$

$\qquad B,\ x_2 y_2$

with an invariant-factor u^2.

Again $\qquad A,\ x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3,$

$\qquad B,\ x_2 y_3 + x_3 y_2$

with two invariant-factors u^2, u^2. But if we write

$$X_1 = x_1 + x_4,\ X_2 = x_2 + x_3,\ X_3 = x_2 - x_3,\ X_4 = x_1 - x_4,$$

the parts in A, B, corresponding to the two invariant-factors, can be separated, thus,*

$$A,\ \tfrac{1}{2}\left[(X_1 Y_2 + X_2 Y_1) + (X_3 Y_4 + X_4 Y_3)\right]$$
$$B,\ \tfrac{1}{2}(X_2 Y_2 - X_3 Y_3).$$

(iii). $(G_2 L_2)$ and $(L_2 K_1)$:

$\qquad A,\ x_1 y_2 + x_2 y_1$

$\qquad B,\ x_2 y_3 + x_3 y_2$

giving $\qquad |uA + vB| \equiv 0$

The general forms obtained from the continuation of the process given for $(G_2 L_3)$ and $(L_3 M_1)$ above (p. 245) will be:

First, $\qquad A,\ x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1}$

$\qquad B - cA,\ x_2 y_3 + x_3 y_2 + \ldots + x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2}.$

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* See, in particular, Kronecker, Berliner Monatsberichte, 1874, p. 59 = Ges. Werke, vol. I, p. 351.

corresponding to two invariant-factors $(u+cv)^m$, $(u+cv)^m$ of $|uA+vB|$. The parts corresponding to each factor can be separated by writing

$$X_r = x_r + x_{2m+1-r},\ X_{2m+1-r} = x_r - x_{2m+1-r},$$
$$(r = 1, 2, \ldots, m).$$

Then if m be odd we find a pair of terms in the middle of A of the form $\frac{1}{2}(X_m Y_m - X_{m+1} Y_{m+1})$ and the general form is

$$\begin{aligned} A,\quad & \tfrac{1}{2}(X_1 Y_2 + X_2 Y_1 + X_3 Y_4 + X_4 Y_3 + \ldots + X_m Y_m), \\ & + \tfrac{1}{2}(-X_{m+1} Y_{m+1} + X_{m+2} Y_{m+3} + X_{m+3} Y_{m+2} + \ldots \\ & \qquad + X_{2m-1} Y_{2m} + X_{2m} Y_{2m-1}), \\ B - cA,\quad & \tfrac{1}{2}(X_2 Y_3 + X_3 Y_2 + \ldots + X_{m-1} Y_m + X_m Y_{m-1}), \\ & + \tfrac{1}{2}(X_{m+1} Y_{m+2} + X_{m+2} Y_{m+1} + \ldots + X_{2m-2} Y_{2m-1} + X_{2m-1} Y_{2m-2}). \end{aligned}$$

But if m be even, we find this pair of terms $\frac{1}{2}(X_m Y_m - X_{m+1} Y_{m+1})$ in the middle of $(B - cA)$ and the forms are,

$$\begin{aligned} A,\quad & \tfrac{1}{2}(X_1 Y_2 + X_2 Y_1 + \ldots + X_{m-1} Y_m + X_m Y_{m-1}), \\ & + \tfrac{1}{2}(X_{m+1} Y_{m+2} + X_{m+2} Y_{m+1} + \ldots + X_{2m-1} Y_{2m} + X_{2m} Y_{2m-1}), \\ B - cA,\quad & \tfrac{1}{2}(X_2 Y_3 + X_3 Y_2 + \ldots + X_m Y_m), \\ & + \tfrac{1}{2}(-X_{m+1} Y_{m+1} + X_{m+2} Y_{m+3} + X_{m+3} Y_{m+2} + \ldots \\ & \qquad + X_{2m-2} Y_{2m-1} + X_{2m-1} Y_{2m-2}).* \end{aligned}$$

Second
$$\begin{aligned} A,\quad & x_2 y_3 + x_3 y_2 + \ldots + x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2} \\ B,\quad & x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1} \end{aligned}$$

corresponding to the invariant-factors v^m, v^m of $|uA+vB|$. The parts can be separated as in the first case.

Third,
$$\begin{aligned} A,\quad & x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1} + x_{2m+1} y_{2m+1}, \\ B - cA,\quad & x_2 y_3 + x_3 y_2 + \ldots + x_{2m} y_{2m+1} + x_{2m+1} y_{2m} \end{aligned}$$

corresponding to the single invariant-factor $(u+cv)^{2m+1}$ of $|uA+vB|$.

Fourth,
$$\begin{aligned} A,\quad & x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1}, \\ B - cA,\quad & x_2 y_3 + x_3 y_2 + \ldots + x_{2m} y_{2m}, \end{aligned}$$

corresponding to the single invariant-factor $(u+cv)^{2m}$.

The *fifth* and *sixth* cases correspond to the invariant-factors v^{2m+2}, v^{2m} and are obtained from the third and fourth by putting $c = 0$ and interchanging A, B.

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*Kronecker states in the paper just quoted (Ges. W. I. p. 367) that this division into two parts is only possible if $m = 2n$ or is *even*; but apparently this is an oversight, as on p. 354 (c) he makes no restriction on m.

Seventh,

$$A, \quad x_1 y_2 + x_2 y_1 + \dots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1},$$
$$B, \quad x_2 y_3 + x_3 y_2 + \dots + x_{2m} y_{2m+1} + x_{2m+1} y_{2m}$$

corresponding to $|uA + vB| \equiv 0$. Here we have obtained no rule for finding m directly from the determinant without carrying out the reduction (see footnote to the next class of reductions).

A Symmetric and an Alternate Form.

The simple sets of terms first found will give types as below:

(i). (G_1): (terms belonging to the symmetric form only),

$$A, \quad x_1y_1 \text{ or } x_1y_2 + x_2y_1.$$

In the first case, we have an invariant-factor u of $|uA + vB|$; in the second, two u, u. The second case is equivalent to two of the first, by writing it in the form

$$\tfrac{1}{2}\left[(x_1 + x_2)(y_1 + y_2) - (x_1 - x_2)(y_1 - y_2)\right].$$

Terms belonging to the alternate form only:

$$B, \quad x_1y_2 - x_2y_1.$$

Here we have two invariant-factors v, v, and the form *cannot* be split up into two, one for each invariant-factor.

(ii). (G_2L_1) and (L_1):

$$A, \quad x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3,$$
$$B, \quad x_2y_3 - x_3y_2.$$

Here we have two invariant-factors u^2, u^2 which cannot be separated Again we may have, interchanging the parts played by A and B,

$$A, \quad x_2y_3 + x_3y_2,$$
$$B, \quad x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3,$$

with two invariant-factors v^2, v^2; these can be separated by writing

$$x_1 + x_4 = X_1, \quad x_1 - x_4 = X_4, \quad x_2 - x_3 = X_2, \quad x_2 + x_3 = X_3,$$

with the corresponding substitutions for the y's. Then the types become

$$A, \quad \tfrac{1}{2}\left[-X_2Y_2 + X_3Y_3\right],$$
$$B, \quad \tfrac{1}{2}\left[X_1Y_2 - X_2Y_1 + Y_3X_4 - Y_4X_3\right],$$

in which the parts are separable. Finally, we may have

$$\begin{aligned} &A, \quad x_2 y_2, \\ &B, \quad x_1 y_2 - x_2 y_1, \end{aligned}$$

with an invariant-factor v^2.

(iii). $(G_2 L_2)$ and $(L_2 K_1)$:

$$\begin{aligned} &A, \quad x_1 y_2 + x_2 y_1, \\ &B, \quad x_2 y_3 - x_3 y_2, \end{aligned}$$

and here $|uA + vB| \equiv 0$ for all values of u, v.

This exhausts all the possibilities for the specially simple types; we proceed to examine the more general forms, of which the foregoing are particular cases. By continuing the process indicated above for dealing with $(G_2 L_3)$ and $(L_3 M_1)$, we obtain the following possible cases:

First:
$$\begin{aligned} &A, \quad x_1 y_2 + x_2 y_1 + \dots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1}, \\ &B, \quad x_2 y_3 - x_3 y_2 + \dots + x_{2m-2} y_{2m-1} - x_{2m-1} y_{2m-2}, \end{aligned}$$

a case which corresponds to two invariant-factors u^m, u^m of $|uA + vB|$; if m be odd, the parts corresponding to the two may be separated, but not if m be even* (the reason being that there are $(m-1)$ pairs of terms in B, and when m is odd, these can be divided into two sets each containing $\frac{1}{2}(m-1)$ pairs).

Second:
$$\begin{aligned} &A, \quad x_2 y_3 + x_3 y_2 + \dots + x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2}, \\ &B, \quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1}, \end{aligned}$$

corresponding to the factors v^m, v^m of $|uA + vB|$; if m be even, the two parts can be separated, but not if m be odd (for here there are m pairs in B). The separating substitutions are analogous to those required for the last case.

Third:
$$\begin{aligned} &(A), \quad (x_1 y_2 + x_2 y_1) + \dots + (x_{2m-2} y_{2m} + x_{2m} y_{2m-1}) + x_{2m+1} y_{2m+1}, \\ &(B), \quad (x_2 y_2 - x_3 y_2) + \dots + (x_{2m} y_{2m+1} - x_{2m+1} y_{2m}), \end{aligned}$$

corresponding to the single invariant-factor u^{2m+1} of $|uA + vB|$.

Fourth:
$$\begin{aligned} &(A), \quad (x_2 y_3 + x_3 y_2) + \dots + (x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2}) + x_{2m} y_{2m}, \\ &(B), \quad (x_1 y_2 - x_2 y_1) + \dots + (x_{2m-1} y_{2m} - x_{2m} y_{2m-1}), \end{aligned}$$

---

* If we write

$$\left.\begin{aligned} \xi_k &= x_k - (-1)^{m-k} x_{2m+1-k} \\ \xi_{m+k} &= x_k + (-1)^{m-k} x_{2m+1-k} \end{aligned}\right\}, \quad (k = 1, 2, \dots, m).$$

and make the congruent substitutions for the η's in term of the y's, we find that A, B divide into the parts

$$\begin{aligned} &A, \quad [\tfrac{1}{2}(\xi_1\eta_2 + \xi_2\eta_1) + \dots + \tfrac{1}{2}\xi_m\eta_m] + [\tfrac{1}{2}(\xi_{m+1}\eta_{m+2} + \xi_{m+2}\eta_{m+1}) + \dots - \tfrac{1}{2}\xi_{2m}\eta_{2m}], \\ &B, \quad [\tfrac{1}{2}(\xi_2\eta_3 - \xi_3\eta_2) + \dots + \tfrac{1}{2}(\xi_{m-1}\eta_m - \xi_m\eta_{m-1})] + [\tfrac{1}{2}(\xi_{m+2}\eta_{m+3} - \xi_{m+3}\eta_{m+2}) + \dots + \tfrac{1}{2}(\xi_{2m-1}\eta_{2m} - \xi_{2m}\eta_{2m-1})]. \end{aligned}$$

for the single invariant-factor v^{2m} of $|uA+vB|$.

$$\begin{array}{lll} \textit{Fifth}: & (A), & (x_1y_2+x_2y_1)+\ldots\ldots+(x_{2m-1}y_{2m}+x_{2m}y_{2m-1}), \\ & (B), & (x_2y_3-x_3y_2)+\ldots\ldots+(x_{2m}y_{2m+1}-x_{2m+1}y_{2m}); \end{array}$$

here $|uA+vB|\equiv 0$*.

We have now exhausted all the types that can be found by the first method of investigation indicated above; the complete reduction, corresponding to non-zero roots of $|\lambda A-B|=0$, has to be effected in a different way, as already explained. Thus we find the type of reduced terms:

$$\begin{array}{lll} \textit{Sixth}: & (A), & (x_1y_2+x_2y_1)+\ldots\ldots+(x_{2s-1}y_{2s}+x_{2s}y_{2s-1}), \\ & (B), & c\,(x_1y_2-x_2y_1)+\ldots\ldots+c\,(x_{2s-1}y_{2s}-x_{2s}y_{2s-1}) \\ & & +(x_2y_3-x_3y_2)+\ldots\ldots+(x_{2s-2}y_{2s-1}-x_{2s-1}y_{2s-2}), \end{array}$$

corresponding to the pair of invariant factors $(\lambda-c)^s$, $(\lambda+c)^s$ of $|\lambda A-B|$.

Two Alternate Forms.

The simple types are here:

(i). (G): $\qquad A, \quad x_1y_2-x_2y_1,$

with two invariant-factors u, u of $|uA+vB|$.

(ii). (G_2L_1) and (L_1):

$$\begin{array}{ll} A, & x_1y_2-x_2y_1+x_3y_4-x_4y_3 \\ B, & x_2y_3-x_3y_2, \end{array}$$

giving two invariant-factors u^2, u^2.

(iii). (G_2L_2) and (L_2K_1):

$$\begin{array}{ll} A, & x_1y_2-x_2y_1, \\ B, & x_2y_3-x_3y_2, \end{array}$$

corresponding to the simplest types that give $|uA+vB|\equiv 0$.

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*It is one drawback to the method explained here, that the number m cannot be determined (so far as we have shown) from the determinant $|uA+vB|$ and its minors. According to the foregoing, it would seem to be necessary to calculate the reduced form before we can find m, but this is not the case, as will be seen from Kronecker's paper and those quoted above. I shall not give the rules for its determination, which will be found in Muth's "Elementartheiler" (p. 108), where m is called a "Minimalgradzahl." Note that, in all the problems examined in this paper, the two series of m's given by Muth are necessarily the same.

We have only to add the results obtained by continuing the process already indicated for dealing with $(G_2 L_3)$ and $(L_3 M_1)$; we obtain a smaller number of types than in the case of a symmetric and an alternate form.
They are, in order:

First, $\quad A, \quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1},$
$\quad B - cA, \quad x_2 y_3 - x_3 y_2 + \dots + x_{2m-2} y_{2m-1} - x_{2m-1} y_{2m-2},$

corresponding to the two invariant-factors $(u+cv)^m$, $(u+cv)^m$, of $|uA+vB|$. (Here c may be zero.)

Second, $\quad A, \quad x_2 y_3 - x_3 y_2 + \dots + x_{2m-2} y_{2m-1} - x_{2m-1} y_{2m-2},$
$\quad B, \quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1},$

corresponding to the two invariant-factors v^m, v^m, of $|uA+vB|$.

Third, $\quad A, \quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1},$
$\quad B, \quad x_2 y_3 - x_3 y_2 + \dots + x_{2m} y_{2m+1} - x_{2m+1} y_{2m},$

corresponding to $|uA+vB| \equiv 0$. (For the meaning of m see the last footnote.)

It will be observed that here the invariant-factors *always* occur in pairs; and not, as in the case of a symmetric and an alternate form, sometimes singly.

As an illustration of the methods explained we may take the two forms,

$$A = a(x_2 y_3 - x_3 y_2) + b(x_3 y_1 - x_1 y_3) + c(x_1 y_2 - x_2 y_1) + p(x_1 y_4 - x_4 y_1) + q(x_2 y_4 - x_4 y_2) + r(x_3 y_4 - x_4 y_3).$$

$B = x_1 y_1 + x_2 y_2 + x_3 y_3.$

Let us write
$$-\xi_4 = \frac{\partial A}{\partial y_3} = -rx_4 - bx_1 + ax_2,$$
$$\xi_3 = \frac{\partial A}{\partial y_4} = px_1 + qx_2 + rx_3,$$

then, as in §1, we have
$$A = \xi_3 \eta_4 - \xi_4 \eta_3 + \frac{\theta}{r}(x_1 y_2 - x_2 y_1),$$
where
$$\theta = ap + bq + cr,$$

We may note that θ is the Pfaffian of $|A|$. Substituting in B for x_3, y_3 in terms of ξ_3, η_3 we have

$$B = x_1 y_1 + x_2 y_2 + \frac{1}{r^2}[\xi_3 - (px_1 + qx_2)][\eta_3 - (py_1 + qy_2)];$$

if we apply §2 to B we find that with,

$$\xi_1 = x_1 - \frac{p}{\sigma^2}\xi_3, \quad \xi_2 = x_2 - \frac{q}{\sigma^2}\xi_3,$$
$$\sigma^2 = p^2 + q^2 + r^2,$$

we have

$$B = \xi_1\eta_1 + \xi_2\eta_2 + \frac{1}{r^2}(p\xi_1 + q\xi_2)(p\eta_1 + q\eta_2) + \frac{1}{\sigma^2}\xi_3\eta_3$$

Substitute in A for x_1, y_1, x_2, y_2, in terms of $\xi_1, \eta_1, \xi_2, \eta_2$ and then

$$A = \xi_3 Y_4 - \eta_3 X_4 + \frac{\theta}{r}(\xi_1\eta_2 - \xi_2\eta_1),$$

where $$X_4 = \xi_4 + \frac{\theta}{r\sigma^2}(p\xi_2 - q\xi_1),$$

Turning again to B, we have, if

$$\bar{\xi}_1 = \xi_1 + \frac{pq}{p^2 + r^2}\xi_2$$

$$B = \frac{p^2 + r^2}{r^2}\bar{\xi}_1\bar{\eta}_1 + \frac{\sigma^2}{p^2 + r^2}\xi_2\eta_2 + \frac{1}{\sigma^2}\xi_3\eta_3$$

and this substitution does not alter the form of A.

Finally write,

$$X_1 = (p^2 + r^2)^{\frac{1}{2}}\bar{\xi}_1 / r,\; X_2 = \sigma\xi_2 / (p^2 + r^2)^{\frac{1}{2}},\; X_3 = \xi_3/\sigma,$$

and then we have

$$A = \sigma(X_3 Y_4 - X_4 Y_3) + \frac{\theta}{\sigma}(X_1 Y_2 - X_2 Y_1),$$

$$B = X_1 Y_1 + X_2 Y_2 + X_3 Y_3,$$

It will be observed that this form of reduction is by no means unique. For we may obviously take instead of X_4 the linear function $(X_4 + \alpha X_3)$, where α is arbitrary; and in place of X_1 and X_2, $X_1 \cos\beta - X_2 \sin\beta$, and $X_1 \sin\beta + X_2 \cos\beta$, where β is also arbitrary but real. We can make the function $(X_4 + \alpha X_3)$ symmetrical in x_1, x_2, x_3 by writing $\alpha = (aq - bp)/r\sigma^2$.

This agrees with what we know from the geometrical interpretation; for the problem is the reduction of a linear complex to its central axis.

5.—*Hermite's Forms.*

These are bilinear forms such as $\Sigma a_{rs} x_r y_s$, in which a_{rs}, a_{sr} are conjugate imaginaries (in particular a_{rr} is real), and x_r, y_r are also conjugate imaginaries. The methods explained above can be applied to the reduction of a pair of Hermite's forms, A, B, in which the substitutions on the x's and y's are conjugate imaginaries.

For consider the methods of §1; if $a_{11} \neq 0$ we have, as there shewn

$$A = \frac{1}{a_{11}} \frac{\partial A}{\partial x_1} \frac{\partial A}{\partial y_1} + \frac{1}{a_{11}^2} B,$$

where B is a Hermite's form in $2(n-1)$ variables. But if we write

$$X_1 = \frac{\partial A}{\partial y_1} = a_{11}x_1 + a_{21}x_2 + \ldots + a_{n1}x_n,$$

$$Y_1 = \frac{\partial A}{\partial x_1} = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n,$$

we see by the definitions of the coefficients that these two substitutions are conjugate imaginaries, and then

$$A = \frac{1}{a_{11}} X_1 Y_1 + \frac{1}{a_{11}^2} B.$$

Again if $a_{11} = 0$, we find, as in §1, that

$$A = \frac{1}{a_{12}} \frac{\partial A}{\partial x_1} \frac{\partial A}{\partial y_1} + \frac{1}{a_{21}} \frac{\partial A}{\partial x_2} \frac{\partial A}{\partial y_1} - \frac{a_{22}}{a_{12}a_{21}} \frac{\partial A}{\partial x_1} \frac{\partial A}{\partial y_1} - \frac{1}{a_{12}a_{21}} B,$$

where B is a Hermite's form in $(n-2)$ variables. But, since a_{12}, a_{21} are conjugate imaginaries their product is real and positive; thus if we write

$$X_1 = \frac{1}{a_{12}} \frac{\partial A}{\partial y_2} - \tfrac{1}{2} \frac{a_{22}}{a_{12}a_{21}} \frac{\partial A}{\partial y_1}, \quad X_2 = \frac{\partial A}{\partial y_1},$$

$$Y_1 = \frac{1}{a_{21}} \frac{\partial A}{\partial x_2} - \tfrac{1}{2} \frac{a_{22}}{a_{12}a_{21}} \frac{\partial A}{\partial x_1}, \quad Y_2 = \frac{\partial A}{\partial x_1},$$

the substitutions are conjugate imaginaries, for a_{22} is real. Then

$$A = X_1 Y_2 + X_2 Y_1 - \frac{1}{a_{12}a_{21}} B.$$

Passing to the methods of §2, we must first see that the minor D_{11} is real; to prove this we note that the change of $+i$ to $-i$ in D_{11} will only change rows into columns and so will not alter D_{11}. By a similar argument the minors D_{12} D_{21} are conjugate imaginaries.

In the first place, if $D_{11} \neq 0$ we have

$$A = \frac{D}{D_{11}} x_1 y_1 + C,$$

where C is derived from A by writing zero for x_1, y_1 and X_r, Y_r for x_r, y_r $(r=2, 3, \ldots, n)$. Further X_r, Y_r are defined by

$$\frac{\partial C}{\partial X_r} = \frac{\partial A}{\partial x_r}, \frac{\partial C}{\partial Y_r} = \frac{\partial A}{\partial y_r} \quad (r=2, 3, \ldots, n),$$

from which it readily follows that the substitutions are conjugate imaginaries.

Again, if $D_{11}=0$, $(D \neq 0)$, we find

$$A = \frac{D}{D_{12}} x_1 y_2 + \frac{D}{D_{21}} x_2 y_1 - \frac{DD_{22}}{D_{12}D_{21}} x_1 y_1 + C,$$

where C is found by writing in A zero for x_1, y_1, x_2, y_2 and X_r, Y_r for x_r, y_r, $(r=3, 4, \ldots, n)$.

If now
$$X_2 = \frac{D}{D_{21}} x_2 - \tfrac{1}{2} \frac{DD_{22}}{D_{12}D_{21}} x_1,$$
$$Y_2 = \frac{D}{D_{12}} y_2 - \tfrac{1}{2} \frac{DD_{22}}{D_{12}D_{21}} y_1,$$
we have
$$A = x_1 Y_2 + x_2 Y_1 + C,$$
and X_2, Y_2 are conjugate imaginaries; as before, X_r, Y_r are conjugate imaginaries.

Thus, if $|A|=0$ or $|B|=0$, we can reduce a pair of Hermite's forms (A, B) by a process analogous to that given before for symmetric forms (or quadratics); also, if $\lambda = c$ be a *real* root of $|\lambda A - B| = 0$, the same method can be applied, for $(cA - B)$ is then a Hermite's form. But, in general, some of the roots of $|\lambda A - B|$ will not be real; and if c be complex, $(cA - B)$ is no longer a Hermite's form. Thus, our general process of reduction fails for these complex roots; and, to carry out the reduction, we must proceed as in the case of a symmetric and an alternate form above. There are certain obvious changes, but the reductions can be arranged to correspond step by step; the final typical reduced sets of terms being

$$\begin{aligned} &(A), \quad (x_1 y_2 + x_2 y_1) + \cdots + (x_{2s-1} y_{2s} + x_{2s} y_{2s-1}), \\ &(B), \quad (c x_1 y_2 + c_0 x_2 y_1) + \cdots + (c x_{2s-1} y_{2s} + c_0 x_{2s} y_{2s-1}) \\ &\qquad + (x_2 y_3 + x_3 y_2) + \cdots + (x_{2s-2} y_{2s-1} + x_{2s-1} y_{2s-2}), \end{aligned}$$

corresponding to the pair of invariant-factors $(\lambda - c)^s$, $(\lambda - c_0)^s$, c_0 being the conjugate imaginary to c. The other types are precisely the same as those for symmetric forms and will not be repeated now.

ST. JOHN'S COLLEGE, CAMBRIDGE (ENG.), *Jan.* 1, 1901.

On the Imprimitive Substitution Groups of Degree Fifteen and the Primitive Substitution Groups of Degree Eighteen.

By Emilie Norton Martin.

The following work is, with some slight modifications, the same as that of which an abstract was presented at the summer meeting of the American Mathematical Society in 1899. With regard to the imprimitive groups of degree fifteen, which form the subject matter of the first part of this paper, it should be stated that I have added two new groups to the list as originally presented, namely, the groups with five systems of imprimitivity simply isomorphic to the alternating and symmetric groups of degree 5, and that Dr. Kuhn reported at the February meeting of the Society, 1900, that he had carried the investigation further, adding 28 to the 70 groups that I succeeded in finding.

In the second part of this paper the determination of the primitive groups of degree 18 depends to a great extent upon the lists of transitive groups of lower degrees already determined. Any new discovery of groups of degree less than 18 would necessitate an examination of such groups to determine whether they can be combined with others in such a way as to generate a primitive group of degree 18. This list, therefore, cannot claim to be absolutely complete, since omissions are always possible.

Imprimitive Substitution Groups of Degree Fifteen.

Every imprimitive group contains a self-conjugate intransitive subgroup consisting of all the operations that interchange the elements of the systems of imprimitivity among themselves without interchanging the systems. Therefore, the problem of the determination of all imprimitive groups of degree 15 falls into two parts: 1st, the determination of all intransitive groups of degree 15

capable of becoming the self-conjugate subgroups of such imprimitive groups; 2d, the determination of substitutions that will interchange the systems of imprimitivity and at the same time fulfill other conditions depending upon the particular group under discussion. The intransitive self-conjugate subgroup is called for shortness the *head*, the remaining substitutions of the imprimitive group are designated as the *tail*, a terminology that has been adopted by Dr. G. A. Miller in his papers on imprimitive groups.

The elements of an imprimitive group of degree 15 may fall into three systems of five elements each, or into five systems of three each. For the first of these cases, certain theorems given by Dr. G. A. Miller (Quar. Jour. Math., vol. XXVIII, 1896) are useful. With a slight modification in notation in order to adapt them to the notation of this paper, they are as follows, where G^1 represents a group in the elements with index 1, while G^2 and G^3 represent precisely the same group in the elements with indices 2 and 3.

THEOREM I.—*All the substitutions that can be used to construct tails are*

$$(a_1^1a_2^1 \ldots\ldots a_n^1) \text{ all } (a_1^2a_2^2 \ldots\ldots a_n^2) \text{ all } (a_1^3a_2^3 \ldots\ldots a_n^3) \text{ all}$$
$$\{(a_1^1a_1^2a_1^3 . a_2^1a_2^2a_2^3 . \ldots\ldots . a_n^1a_n^2a_n^3),\ (a_1^1a_1^2 . a_2^1a_2^2 . \ldots\ldots . a_n^1a_n^2)\}$$
$$-(a_1^1a_2^1 \ldots\ldots a_n^1) \text{ all } (a_1^2a_2^2 \ldots\ldots a_n^2) \text{ all } (a_1^3a_2^3 \ldots\ldots a_n^3) \text{ all.}$$

THEOREM II.—*If* $G^1 = (a_1^1a_2^1 \ldots\ldots a_n^1)$ all, *there are three imprimitive groups with the common head* $(G^1G^2G^3)$ pos, *and two with the common head* G^1 pos G^2 pos G^3 pos $+$ G^1 neg G^2 neg G^3 neg.

THEOREM III.—*If* $G^1 = (a_1^1a_2^1 \ldots\ldots a_n^1)$ pos, *there are three imprimitive groups with the common head* $G^1G^2G^3$, *and three with the common head* $(G^1G^2G^3)_{1,1,1}$.

THEOREM IV.—*If the head is* $G^1G^2G^3$, *there is only one group which corresponds to* (abc) cyc.

The possible heads for these groups are got either by the direct multiplication of transitive groups of degree 5 in the three systems of elements, or by the establishment of isomorphic relations between such groups.

The transitive groups of degree 5 are five in number, and fall naturally into two categories, the first containing the symmetric group and its self-conjugate subgroup, the alternating group, the second containing the metacyclic group, together with its two self-conjugate transitive subgroups. These five groups are represented respectively by

$$(a_1a_2a_3a_4a_5) \text{ all, } (a_1a_2a_3a_4a_5) \text{ pos, } (a_1a_2a_3a_4a_5)_{20},\ (a_1a_2a_3a_4a_5)_{10},\ (a_1a_2a_3a_4a_5)_5 .$$

From the first two groups come the following heads:

$$\begin{array}{rl}
\text{I.} & (a_1^1a_2^1a_3^1a_4^1a_5^1) \text{ all } (a_1^2a_2^2a_3^2a_4^2a_5^2) \text{ all } (a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ all} = H_{1728000}. \\
\text{II.} & \{(a_1^1a_2^1a_3^1a_4^1a_5^1) \text{ all } (a_1^2a_2^2a_3^2a_4^2a_5^2) \text{ all } (a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ all}\} \text{ pos} = H_{864000}. \\
\text{III.} & (a_1^1a_2^1a_3^1a_4^1a_5^1) \text{ pos } (a_1^2a_2^2a_3^2a_4^2a_5^2) \text{ pos } (a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ pos} \\
 & + (a_1^1a_2^1a_3^1a_4^1a_5^1) \text{ neg } (a_1^2a_2^2a_3^2a_4^2a_5^2) \text{ neg } (a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ neg} = H_{432000}. \\
\text{IV.} & (a_1^1a_2^1a_3^1a_4^1a_5^1) \text{ pos } (a_1^2a_2^2a_3^2a_4^2a_5^2) \text{ pos } (a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ pos} = H_{216000}. \\
\text{V.} & (a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ all} = H_{120}. \\
\text{VI.} & (a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ pos} = H_{60}.
\end{array}$$

From the three remaining groups come the heads :

$$\begin{array}{rl}
\text{VII.} & (a_1^1a_2^1a_3^1a_4^1a_5^1)_{20} (a_1^2a_2^2a_3^2a_4^2a_5^2)_{20} (a_1^3a_2^3a_3^3a_4^3a_5^3)_{20} = H_{8000}. \\
\text{VIII.} & \{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20} (a_1^2a_2^2a_3^2a_4^2a_5^2)_{20} (a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\} \text{ pos} = H_{4000}. \\
\text{IX.} & \{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20} (a_1^2a_2^2a_3^2a_4^2a_5^2)_{20} (a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}_{10,\,10,\,10} = H_{2000}. \\
\text{X.} & (a_1^1a_2^1a_3^1a_4^1a_5^1)_{10} (a_1^2a_2^2a_3^2a_4^2a_5^2)_{10} (a_1^3a_2^3a_3^3a_4^3a_5^3)_{10} = H_{1000}. \\
\text{XI.} & \{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20} (a_1^2a_2^2a_3^2a_4^2a_5^2)_{20} (a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}_{5,\,5,\,5} = H_{500}. \\
\text{XII.} & \{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{10} (a_1^2a_2^2a_3^2a_4^2a_5^2)_{10} (a_1^3a_2^3a_3^3a_4^3a_5^3)_{10}\}_{5,\,5,\,5} = H_{250}. \\
\text{XIII.} & (a_1^1a_2^1a_3^1a_4^1a_5^1) \text{ cyc } (a_1^2a_2^2a_3^2a_4^2a_5^2) \text{ cyc } (a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ cyc} = H_{125}. \\
\text{XIV.} & (a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)_{20} = H_{20}. \\
\text{XV.} & (a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3)_{10} = H_{10}. \\
\text{XVI.} & (a_1^1a_2^1a_3^1a_4^1a_5^1 \,.\, a_1^2a_2^2a_3^2a_4^2a_5^2 \,.\, a_1^3a_2^3a_3^3a_4^3a_5^3) \text{ cyc.} = H_5.
\end{array}$$

The groups corresponding to these heads may be isomorphic either to $(a^1a^2a^3)$ cyc or to $(a^1a^2a^3)$ all. To generate a group isomorphic to $(a^1a^2a^3)$ cyc a substitution with the following properties must be added to the head: it must have its cube in the head, it must interchange all three systems, and it must transform the head into itself. Calling the group so found G, the groups isomorphic to $(a^1a^2a^3)$ all may be found by combining with G any substitution that has its square in the head, that interchanges two of the systems leaving the third unaffected, and that transforms H into itself, and G into itself.

As all the heads given above are symmetric in the three sets of elements, each head furnishes two groups by means of the symmetrically formed substitutions

$$s = a_1^1a_1^2a_1^3 \,.\, a_2^1a_2^2a_2^3 \,.\, a_3^1a_3^2a_3^3 \,.\, a_4^1a_4^2a_4^3 \,.\, a_5^1a_5^2a_5^3\,, \quad t = a_1^1a_1^2 \,.\, a_2^1a_2^2 \,.\, a_3^1a_3^2 \,.\, a_4^1a_4^2 \,.\, a_5^1a_5^2 \,.$$

The letters s and t are used throughout this section of the paper to denote these particular substitutions, other substitutions fulfilling the same conditions being denoted by the same letters with suffixes.

According to Theorem I, any s_a or t_a must be the product of some substitution, σ_a, of the most general head, $H_{1728000}$, by s or t. Therefore σ_a must be a substitution of a subgroup of $H_{1728000}$ that contains the special H under consideration as a self-conjugate subgroup.

We may now proceed to the determination of the groups to be derived from the various heads taken in order.

I. $H_{1728000}$ gives us, according to Theorem I, only the two groups,

$$\{H_{1728000},\ s,\} \text{ of order } 5184000_1,$$
$$\text{and } \{H_{1728000},\ s,\ t\} \text{ of order } 10368000.$$

II. H_{864000} gives us, in accordance with Theorem II, three distinct groups. Of these, two are the groups,

$$\{H_{864000},\ s\} \text{ of order } 2592000_1,$$
$$\{H_{864000},\ s,\ t\} \text{ of order } 5184000_2,$$

A σ that transforms the head into itself without belonging in the head is $\sigma = a_1^1 a_2^1$. This cannot be combined with s, as $(\sigma s)^3$ is an odd substitution; it may, however, be combined with t. The remaining group is therefore

$$\{H_{864000},\ s,\ a_1^1 a_2^1 \,.\, t\} \text{ of order } 5184000_3.$$

Of these two groups of order 5184000, the first contains both odd and even substitutions, the second only even.

III. H_{432000} gives, by Theorem II, the two groups

$$\{H_{432000},\ s\} \text{ of order } 1296000_1,$$
$$\{H_{432000},\ s,\ t\} \text{ of order } 2592000_2.$$

IV. H_{216000} gives us, by Theorem III, three distinct groups. $\sigma = a_1^1 a_2^1$ transforms the head into itself, but when combined with s it gives an odd substitution whose cube cannot be found in the head. The substitution σt furnishes us however, with a new t_a. The three groups are, therefore,

$$\{H_{216000},\ s\} \text{ of order } 648000,$$
$$\{H_{216000},\ s,\ t\} \text{ of order } 1296000_2,$$
$$\{H_{216000},\ s,\ a_1^1 a_2^1 \,.\, t\} \text{ of order } 1296000_3.$$

The two groups of order 1296000 are distinct, since the one contains both odd and even substitutions, the other only even.

V. H_{120} is not contained self-conjugately in any larger subgroup of $H_{1728000}$, therefore only the two following groups can be formed from it:

$$\{H_{120}, s\} \text{ of order } 360_1,$$
$$\{H_{120}, s, t\} \text{ of order } 720.$$

VI. H_{60} gives, in accordance with Theorem III, three groups:

$$\{H_{60}, s\} \text{ of order } 180,$$
$$\{H_{60}, s, t\} \text{ of order } 360_2,$$
$$\{H_{60}, s, a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . t\} \text{ of order } 360_3.$$

The last of these groups consists entirely of even substitutions.

The remaining heads are all composed of substitutions of the type

$$v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'}, \tag{1}$$

where $v_{a^1} = a_2^1 a_3^1 a_5^1 a_4^1$, $u_{a^1} = a_1^1 a_2^1 a_3^1 a_4^1 a_5^1$, while v_{a^2}, u_{a^2}, v_{a^3}, u_{a^3} denote the same substitutions written in elements with the indices 2 and 3 respectively. The substitutions v_{a^1}, u_{a^1} generate the metacyclic group in the five elements with index 1, these substitutions being subject to the conditions

$$v_{a^1}^4 = 1, \quad u_{a^1}^5 = 1, \quad u_{a^1}^x v_{a^1}^y = v_{a^1}^y u_{a^1}^{2^y x}.$$

The most general s_a is given by

$$s_a = v_{a^1}^a u_{a^1}^{a_1} v_{a^2}^b u_{a^2}^{b_1} v_{a^3}^c u_{a^3}^{c_1} s. \tag{2}$$

From this we find

$$s_a^3 = v_{a^1}^{a+b+c} u_{a^1}^{\lambda} v_{a^2}^{a+b+c} u_{a^2}^{\mu} v_{a^3}^{a+b+c} u_{a^3}^{\nu}, \tag{3}$$

where

$$\left.\begin{aligned} \lambda &= 2^c(2^b a_1 + b_1) + c_1, \\ \mu &= 2^a(2^c b_1 + c_1) + a_1, \\ \nu &= 2^b(2^a c_1 + a_1) + b_1. \end{aligned}\right\} \tag{4}$$

Transformation of the general substitution (1) by s_a gives us

$$s_a^{-1} v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'} s_a = v_{a^1}^{\gamma} u_{a^1}^{\nu} v_{a^2}^{\alpha} u_{a^2}^{\lambda} v_{a^3}^{\beta} u_{a^3}^{\mu}, \tag{5}$$

where

$$\left.\begin{aligned} \lambda &= a_1 + 2^a \alpha' - 2^{\alpha} a_1, \\ \mu &= b_1 + 2^b \beta' - 2^{\beta} b_1, \\ \nu &= c_1 + 2^c \gamma' - 2^{\gamma} c_1, \end{aligned}\right\} \tag{6}$$

The general substitution of the group $G = \{H, s_a\}$ is

$$T = s_a^x v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'}. \tag{7}$$

The most general t_β is given by

$$t_\beta = v_{a^1}^{a_2} u_{a^1}^{a_3} v_{a^2}^{b_2} u_{a^2}^{b_3} v_{a^3}^{c_2} u_{a^3}^{c_3} t. \tag{8}$$

Upon squaring this substitution, we get

$$t_\beta^2 = v_{a^1}^{a_2+b_2} u_{a^1}^{\lambda} v_{a^2}^{a_2+b_2} u_{a^2}^{\mu} v_{a^3}^{2c_2} u_{a^3}^{\nu}, \tag{9}$$

where

$$\left.\begin{aligned} \lambda &= 2^{b_2} a_3 + b_3, \\ \mu &= 2^{a_2} b_3 + a_3, \\ \nu &= 2^{c_2} c_3 + c_3. \end{aligned}\right\} \tag{10}$$

On transforming the general substitution T by the general t_β, we have, after a straight-forward calculation, the following expression for the case $x = 1$:

$$t_\beta^{-1} T_{x=1} t_\beta = s^2 v_{a^1}^{-a_2+b_2+\beta+a} u_{a^1}^{\lambda} v_{a^2}^{a_2-c_2+\alpha+c} u_{a^2}^{\mu} v_{a^3}^{-b_2+c_2+\gamma+b} u_{a_3}^{\nu}, \tag{11}$$

where

$$\left.\begin{aligned} \lambda &= -\, 2^{b_2-a_2+\beta+a} a_3 + 2^{\beta+b_2} a_1 + 2^{b_2}\beta' + b_3, \\ \mu &= -\, 2^{a_2-c_2+a+c} c_3 + 2^{a+a_2} c_1 + 2^{a_2}\alpha' + a_3, \\ \nu &= -\, 2^{c_2-b_2+\gamma+b} b_3 + 2^{\gamma+c_2} b_1 + 2^{c_2}\gamma' + c_3. \end{aligned}\right\} \tag{12}$$

We may now return to the consideration of special groups.

VII. H_{8000} gives only the two groups formed with s and t, as any σ that might be used is already contained in this head. The groups are, therefore,

$\{H_{8000}, s\}$ of order 24000_1,
$\{H_{8000}, s, t\}$ of order 48000.

VIII. H_{4000} has the general substitution (1) subject to the condition $\alpha + \beta + \gamma \equiv 0 \pmod 2$. From (3), it is evident that s_a is subject to the condition $a + b + c \equiv 0 \pmod 2$. Therefore, s_a is already in the group generated by H_{4000} and by s, and there is only one group isomorphic to $(a^1a^2a^3)$ cyc. We find by (9) that every t_β has its square in the head, and by (11), that every t_β transforms the head into itself, therefore, we may take as a new t_β the simplest substitution for which $a_2 + b_2 + c_2 \equiv 1 \pmod 2$, viz.:

$$v_{a'} t = a_1^1 a_1^2 \,.\, a_2^1 a_3^2 a_3^1 a_5^2 a_5^1 a_4^2 a_4^1 a_2^2.$$

The three groups with this head are, therefore,

$\{H_{4000}, s\}$ of order 12000_1,
$\{H_{4000}, s, t\}$ of order 24000_2,
$\{H_{4000}, s, v_{a'}t\}$ of order 24000_3.

Of these groups the first and third consist of even substitutions, the second of even and odd.

IX. H_{2000} has the general substitution subject to the condition $\alpha \equiv \beta \equiv \gamma$ (mod 2). From (3) and (5), it is plain that every s_a can be used to generate a group of the kind required. The only possible form for the cofactor of s, if it is not to give the group generated by s and the head, is $v_{a^1}^a v_{a^2}^b v_{a^3}^c$, where a, b, c do not fulfill the condition $a \equiv b \equiv c$ (mod 2). The simplest form for such a cofactor, and a form to which all others reduce, is found by making two of the exponents vanish and the third become equal to 1, e. g., $s_1 = v_{a^1} s = a_1^1 a_1^2 a_1^3 \,.\, a_2^1 a_3^2 a_3^3 a_3^1 a_5^2 a_5^3 a_5^1 a_4^2 a_4^3 a_4^1 a_2^2 a_2^3$ Now, $s_1^4 = s_1 \,.\, v_{a^1} v_{a^2} v_{a^3}$ and $s_1^8 = s_1^2 \,.\, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2$; we may, therefore, take s_1^4 as the s. in the place of s_1 and still have the same group. But $s_1^4 = (v_{a^2}^2 v_{a^3}^3)^{-1} s \, (v_{a^2}^2 v_{a^3}^3)_a$ therefore the group we have now found is merely the transformed of the group generated by s with respect to the substitution $v_{a^2}^2 v_{a^3}^3$. Consequently, there is but one group corresponding to the cyclic group of degree three.

If, in addition to the group given by t, we have a group given by t_β, then according to the relations derived from (11), $a_2 \equiv b_2 \equiv c_2$ (mod 2), i. e., the possible values of t_β are already present in the group generated with the help of t. The two imprimitive groups with this head are, therefore, the groups

$$\{H_{2000}, s\} \text{ of order } 6000.$$
$$\{H_{2000}, s, t\} \text{ of order } 12000_2.$$

In this, and all following work, the terms u in the cofactors of s and t are taken as unity, unless the contrary is expressly stated.

X. H_{1000} has its general substitution subject to the condition $\alpha \equiv \beta \equiv \gamma \equiv 0$ (mod 2). By Theorem IV, this head gives only one group isomorphic to (abc) cyc. If, in addition to the substitution t, there is a substitution t_β, the relations satisfied by the exponents of the v's in (11) reduces to $a_2 \equiv b_2 \equiv c_2$ (mod 2). We have, therefore, two distinct groups according as a_2 is even or odd. The three groups with this head are

$$\{H_{1000}, s\} \text{ of order } 3000_1,$$
$$\{H_{1000}, s, t\} \text{ of order } 6000_2,$$
$$\{H_{1000}, s, v_{a^1} v_{a^2} v_{a^3} t\} \text{ of order } 6000_3.$$

XI. H_{500} subjects the general substitution to the conditions $\alpha = \beta = \gamma$, where $\alpha = 0, 1, 2, 3$. Since every substitution s_a satisfies the necessary conditions, the following independent types of s_a must be examined: $v_{a^1} s$, $v_{a^1}^2 s$, $v_{a^1}^3 s$, $v_{a^1} v_{a^2}^2 s$. The

fourth power of these substitutions is in every case the transformed of s with respect to some combination of the v's; therefore, they give nothing new. The possible forms for t_β are derived from the equation easily deducible from (11); $-a_2+b_2\equiv a_2-c_2\equiv -b_2+c_2$ (mod 4), which, taken in conjunction with the limited range of values of a_2, b_2, c_2, gives $a_2=b_2=c_2$. That is, every possible t_β is already included in the group generated by t. This head gives accordingly only the two groups,

$$\{H_{500},\, s\} \text{ of order } 1500_1,$$
$$\{H_{500},\, s,\, t\} \text{ of order } 3000_2.$$

XII. H_{250} subjects the general substitution (1) to the conditions $\alpha=\beta=\gamma\equiv 0$ (mod 2). To determine an s_α, we have from (3) the condition $a+b+c\equiv 0$ (mod 2). An examination of the four apparently distinct types of s_α, v_1^2s, v_1v_2s, $v_1^3v_2s$, $v_1^3v_2^3s$, shows that just as in the last set of groups, these each give a group that can be derived from the group generated by s by means of an easy transformation.

The possible forms t_β must fulfill the conditions, deducible from (11), $-a_2+b_2\equiv -b_2+c_2\equiv -c_2+a_2\equiv 0$ (mod 2) and also $-a_2+b_2\equiv a_2-c_2$ (mod 4). These reduce to the simple condition $a_2=b_2=c_2$, which furnishes the substitution $t_\beta=v_{a^1}v_{a^2}v_{a^3}t$. This head gives therefore the three groups,

$$\{H_{250},\, s\} \text{ of order } 750_1,$$
$$\{H_{250},\, s,\, t\} \text{ of order } 1500_2,$$
$$\{H_{250},\, s,\, v_{a^1}v_{a^2}v_{a^3}\, t\} \text{ of order } 1500_3.$$

The second group alone contains odd substitutions.

XIII. H_{125} gives in accordance with Theorem IV only one group in which the systems are interchanged cyclically. The general substitution of this head is subject to the condition $\alpha=\beta=\gamma=0$. Applying this condition to (9) and (11) we find $a_2=b_2=c_2$, while a_2 lies under the further restriction of being even. Therefore we have in addition to t the substitution,

$$t_\beta=v_{a^1}^2v_{a^2}^2v_{a^3}^2\,t=a_1^1a_1^2.\,a_2^1a_5^2.\,a_3^1a_4^2.\,a_4^1a_3^2.\,a_5^1a_2^2.\,a_2^3a_5^3.\,a_3^3a_4^3.$$

The three groups given by this head are,

$$\{H_{125},\, s\} \text{ of order } 375,$$
$$\{H_{125},\, s,\, t\} \text{ of order } 750_2,$$
$$\{H_{125},\, s,\, v_{a^1}^2\, v_{a^2}^2\, v_{a^3}^2\, t\} \text{ of order } 750_3.$$

XIV. H_{20} imposes upon the exponents of the general term the conditions

$\alpha = \beta = \gamma$, $\alpha' = \beta' = \gamma'$. Making use of this in (5) and (6) we find $2^a\alpha' \equiv 2^b\alpha' \equiv 2^c\alpha'$ (mod 5), which gives at once $a = b = c$. Using this latter equality in the equations that are deduced from (3) and (4) we find $a_1 = b_1 = c_1$ with the single exception of the case $a = 0$, where the equations become indeterminate, being satisfied by every value of a_1, b_1, c_1. An examination of all of the apparently independent sets of value for a_1, b_1, c_1 shows that in every case the group is transformable into that generated by s alone. In order to determine all substitutions t_β we use the equation, derived from (11), $-a_2 + b_2 \equiv a_2 - c_2 \equiv c_2 - b_2$ (mod. 4), from which follows at once $a_2 = b_2 = c_2$. From (12), by making use of the special case $\alpha = \beta = \gamma = 0$, can be derived the relations $-a_3 + b_3 \equiv -c_3 + a_3 \equiv -b + c_3$ (mod. 5); i, e, $a_3 = b_3 = c_3$. The only groups with this head are therefore the two groups,

$$\{H_{20}, s\} \text{ of order } 60_1,$$
$$\{H_{20}, s, t\} \text{ of order } 120.$$

XV. H_{10} has the general term (1) subject to the conditions $\alpha = \beta = \gamma \equiv 0$ (mod 2), $\alpha' = \beta' = \gamma'$. By precisely the same line of argument as that laid down in the preceding case we arrive at the conclusion $a = b = c$, $a_1 = b_1 = c_1$, $a_2 = b_2 = c_2$, $a_3 = b_3 = c_3$. In this work, too, the indeterminate values of a_1, b_1, c_1 require a careful examination that leads to no new group. From this head come, therefore, the three groups,

$$\{H_{10}, s\} \text{ of order } 30_1,$$
$$\{H_{10}, s, t\} \text{ of order } 60_2,$$
$$\{H_{10}, s, v_{a^1} v_{a^2} v_{a^3} t\} \text{ of order } 60_3.$$

Of these three groups the second alone involves odd substitutions.

XVI. H_5 imposes upon the general term the conditions $\alpha = \beta = \gamma = 0$, $\alpha' = \beta' = \gamma'$, By arguments similar to those used in the last two cases, with the further addition of the condition imposed by (3), $a + b + c \equiv 0$ (mod 4), we find $a = b = c = 0$, $a_1 = b_1 = c_1$. In the determination of t_β we see at once from (9) that c_2 must be even, while from (11) we find $a_2 = b_2 = c_2$, and from (12) $a_3 = b_3 = c_3$.

The groups given by this head are as follows:

$$\{H_5, s\} \text{ of order } 15,$$
$$\{H_5, s, t\} \text{ of order } 30_2,$$
$$\{H_5, s, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2 t\} \text{ of order } 30_3.$$

Passing now to the case of five systems of three elements each, there are seven heads considered in this paper, six involving all the systems symmetrically, the remaining head being unity.

I. $(a_1^1 a_2^1 a_3^1)$ all $(a_1^2 a_2^2 a_3^2)$ all $(a_1^3 a_2^3 a_3^3)$ all $(a_1^4 a_2^4 a_3^4)$ all $(a_1^5 a_2^5 a_3^5)$ all $= H_{7776}$,

II. $\{H_{7776}\}$ pos $= H_{3888}$,

III. $\{H_{7776}\}_{3,3,3,3,3} = H_{486}$,

IV. $(a_1^1 a_2^1 a_3^1)$ pos $(a_1^2 a_2^2 a_3^2)$ pos $(a_1^3 a_2^3 a_3^3)$ pos $(a_1^4 a_2^4 a_3^4)$ pos $(a_1^5 a_2^5 a_3^5)$ pos $= H_{243}$,

V. $(a_1^1 a_2^1 a_3^1 . a_1^2 a_2^2 a_3^2 . a_1^3 a_2^3 a_3^3 . a_1^4 a_2^4 a_3^4 . a_1^5 a_2^5 a_3^5)$ all $= H_6$,

VI. $(a_1^1 a_2^1 a_3^1 . a_1^2 a_2^2 a_3^2 . a_1^3 a_2^3 a_3^3 . a_1^4 a_2^4 a_3^4 . a_1^5 a_2^5 a_3^5)$ cyc $= H_3$,

VII. Unity.

Denoting the system with index r by A_r, it is evident that these systems must be interchanged according to the five groups $(A_1A_2A_3A_4A_5)$ cyc, $(A_1A_2A_3A_4A_5)_{10}$, $(A_1A_2A_3A_4A_5)_{20}$, $(A_1A_2A_3A_4A_5)$ pos, $(A_1A_2A_3A_4A_5)$ all.

The order of procedure in each case is as follows:

1. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ cyc, a substitution s must be found that will interchange the systems cyclically, transform the head into itself, and have its fifth power in the head. The imprimitive group so generated may be called G.

2. If the group is to correspond to $(A_1A_2A_3A_4A_5)_{10}$, it must contain G_1 as a self-conjugate subgroup. In addition, therefore, to the s of case 1, a substitution t must be found that will interchange four of the systems in two pairs, as $A_2A_5 . A_3A_4$, while leaving the remaining system unaltered, and that will, at the same time, transform the head into itself and G into itself. This substitution t must also have its square in the head. This imprimitive group shall be called G_2.

3. If the group is to correspond to $(A_1A_2A_3A_4A_5)_{20}$, it must contain both G_1 and G_2 as self-conjugate subgroups. In addition, therefore, to the s of case 1, a substitution u must be found interchanging four of the systems cyclically, according to $A_2A_3A_5A_4$ for instance, transforming G_1 and G_2 into themselves and having its fourth power in the head.

4. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ pos, two substitutions, v and v', must be found corresponding to $A_1A_2A_3$ and $A_1A_4A_5$. These sub-

stitutions must, therefore, each interchange three systems, leaving two unaltered, they must have their cubes in the head, and must transform the head into itself. This group may be called G'.

5. If the group is to correspond to $(A_1 A_2 A_3 A_4 A_5)$ all, two substitutions, w and w', must be found corresponding to $A_1 A_2 A_3 A_4$ and $A_1 A_5$. G' is to be contained in this new group as a self-conjugate subgroup, therefore w and w' must transform the head into itself and G' into itself. The fourth power of w and the square of w' must both be contained in the head.

I. H_{7776} is the largest possible intransitive group with the given systems of intransitivity, and, consequently, only one group with this head corresponds to each of the transitive groups of degree 5. For each of these groups a substitution or pair of substitutions can be found fulfilling all required conditions and involving the elements of the systems symmetrically. A second set could be found only by multiplying this first step by some substitution belonging to the largest group that contains the head self-conjugately without interchanging any of the systems. But this group is the head itself. The required groups are, therefore, the following:

$$\{H_{7776},\ s\} \text{ of order } 38880,$$
$$\{H_{7776},\ s,\ t\} \text{ of order } 77760,$$
$$\{H_{7776},\ s,\ u\} \text{ of order } 155520,$$
$$\{H_{7776},\ v,\ v'\} \text{ of order } 466560_1,$$
$$\{H_{7776},\ w,\ w'\} \text{ of order } 933120,$$

where

$$\begin{aligned}
s &= a_1^1 a_1^2 a_1^3 a_1^4 a_1^5 \,.\, a_2^1 a_2^2 a_2^3 a_2^4 a_2^5 \,.\, a_3^1 a_3^2 a_3^3 a_3^4 a_3^5,\\
t &= a_1^2 a_1^5 \,.\, a_1^3 a_1^4 \,.\, a_2^2 a_2^5 \,.\, a_2^3 a_2^4 \,.\, a_3^2 a_3^5 \,.\, a_3^3 a_3^4,\\
u &= a_1^2 a_1^3 a_1^5 a_1^4 \,.\, a_2^2 a_2^3 a_2^5 a_2^4 \,.\, a_3^2 a_3^3 a_3^5 a_3^4,\\
v &= a_1^1 a_1^2 a_1^3 \,.\, a_2^1 a_2^2 a_2^3 \,.\, a_3^1 a_3^2 a_3^3,\\
v' &= a_1^1 a_1^4 a_1^5 \,.\, a_2^1 a_2^4 a_2^5 \,.\, a_3^1 a_3^4 a_3^5,\\
w &= a_1^1 a_1^2 a_1^3 a_1^4 \,.\, a_2^1 a_2^2 a_2^3 a_2^4 \,.\, a_3^1 a_3^2 a_3^3 a_3^4,\\
w' &= a_1^1 a_1^5 \,.\, a_2^1 a_2^5 \,.\, a_3^1 a_3^5.
\end{aligned}$$

These letters shall be kept throughout this section of the paper to denote these symmetrically formed substitutions, other substitutions with corresponding properties being denoted by the same letters with suffixes.

II. H_{3888} gives only one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ cyc, viz., the group generated by s. Any new s_a must have as cofactor an odd substitution belonging to H_{7776}, but the fifth power of such a substitution is not contained in the head. There are, however, two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$, since both t and $t_a = a_1^1 a_2^1 . t$ fulfill the necessary conditions. The former generates a group G_{38880_2} containing only even substitutions, the latter generates a group G_{38880_3} containing both odd and even substitutions. There are likewise two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{20}$, one generated by u, the other by $a_1^1 a_2^1 . u$. The first of these groups contains odd substitutions, the second only even. G_{38880_2} is contained self-conjugately in both.

Only one group G' can be found for this head, as no new v_a or v'_a fulfills the necessary conditions. Such a substitution would necessarily be of the form σv or $\sigma v'$, where σ would belong to the group H_{7776}. If σ were even, the group so generated would be a repetition of the group generated by v and v'. If σ were odd, the cubes of σv, $\sigma v'$ would not be contained in the head.

Two groups can be found isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ all, the substitutions w and w' generating one group, the substitutions $a_1^1 a_2^1 . w$, $a_1^1 a_2^1 . w'$ generating the other. This latter group contains only even substitutions.

From this head we have, therefore, derived eight groups:

$\{H_{3888}, s\}$ of order 19440,
$\{H_{3888}, s . t\}$ of order 38880_2,
$\{H_{3888}, s, a_1^1 a_2^1 . t\}$ of order 38880_3,
$\{H_{3888}, s, u\}$ of order 76660_2,
$\{H_{3888}, s, a_1^1 a_2^1 . u\}$ of order 76660_3,
$\{H_{3888}, v, v'\}$ of order 233280,
$\{H_{3888}, w, w'\}$ of order 466560_2,
$\{H_{3888}, a_1^1 a_2^1 . w, a_1^1 a_2^1 . w'\}$ of order 466560_3.

III. H_{486} furnishes us with only one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ cyc, for an examination of the groups given by all possible types of substitutions s_a shows that each of these groups is merely the group generated by the help of s and transformed with respect to some easily discovered substitution. Moreover, there is but one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$, viz., that generated with the help of t. Any cofactor of t must be of one of the types $a_1^1 a_2^1$, $a_1^2 a_2^2 . a_1^5 a_2^5$, $a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^5 a_2^5$, $a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5$, but any t_a got by means of these, transforms

s into s^4 (a substitution not in the head). Precisely the same reasoning shows that there is only the one group isomorphic to $(A_1A_2A_3A_4A_5)_{20}$.

In addition to the group isomorphic to $(A_1A_2A_3A_4A_5)$ pos generated by means of the substitutions v and v', we must examine groups generated with the help of v_a and v'_a, substitutions which contain as cofactors of v, v' respectively the products of transposition, one transposition from each system. A number of these may be rejected at once, but we are left with the possible forms:

$$\begin{aligned} v_1 &= a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, v = a_1^1a_2^2a_1^3 \,.\, a_2^1a_1^2a_2^3 \,.\, a_3^1a_3^2a_3^3, \\ v_2 &= a_1^1a_2^1 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5 \,.\, v = a_1^1a_2^2a_2^3 \,.\, a_2^1a_1^2a_1^3 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5, \\ v'_1 &= a_1^1a_2^1 \,.\, a_1^4a_2^4 \,.\, v' = a_1^1a_2^4a_1^5 \,.\, a_2^1a_1^4a_2^5 \,.\, a_3^1a_3^4a_3^5, \\ v'_2 &= a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, v' = a_1^1a_2^4a_2^5 \,.\, a_2^1a_1^4a_1^5 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3. \end{aligned}$$

But, since v_1, v_2^4 are transformable into v, and v'_1, $v_2'^4$ are transformable into v', it is impossible to generate any group by means of any combination of these four substitutions excepting a group that can be transformed into the one generated by means of v and v'.

A similar examination of all groups isomorphic to $(A_1A_2A_3A_4A_5)$ all, shows that, in addition to the group generated with the help of w and w', there is one other group generated by means of $w_a = a_1^5a_2^5 \,.\, w$ and w'.

From this head are therefore formed the six following groups:

$\{H_{486}, s\}$ of order 2430_1,
$\{H_{486}, s, t\}$ of order 4860_1,
$\{H_{486}, s, u\}$ of order 9720,
$\{H_{486}, v, v'\}$ of order 29160_1,
$\{H_{486}, w, w'\}$ of order 58320_1,
$\{H_{436}, a_1^5a_2^5 \,.\, w, w'\}$ of order 58320_2.

IV. H_{243} gives one group isomorphic to $(A_1A_2A_3A_4A_5)$ cyc, by means of s. The only other permissible forms of s_a are of the type

$$s_1 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, s, \quad s_2 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, a_1^4a_2^4 \,.\, s.$$

But s_1 and s_2 are each the transformed of s with respect to some substitution that transforms the head into itself; therefore, there is only the one group of this type. On the other hand, there are two groups isomorphic to $(A_1A_2A_3A_4A_5)_{10}$, since both t and $t_1 = a_1^1a_2^1 \,.\, a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5 \,.\, t$ fulfill all necessary conditions and generate, one a group of even substitutions, the other a group containing odd

substitutions. There are also two groups isomorphic to $(A_1A_2A_3A_4A_5)_{20}$, one containing both odd and even substitutions, the other only even. These are generated respectively by means of u and of $u_1 = a_1^1a_2^1 . a_1^2a_2^2 . a_1^3a_2^3 . a_1^4a_2^4 . a_1^5a_2^5 . u$, and each contains as a self-conjugate subgroup the group isomorphic to $(A_1A_2A_3A_4A_5)_{10}$ that consists entirely of even substitutions.

Only one group can be found isomorphic to $(A_1A_2A_3A_4A_5)$ pos, and this is the one formed by the help of v and v'. An examination of the various substitutions v_α and v'_α corresponding to various types of cofactor of v and v' shows that all groups formed by means of these substitutions are transformable into the one group.

On the other hand, we have two distinct groups corresponding to $(A_1A_2A_3A_4A_5)$ all, the one consisting of both odd and even substitutions and generated by the aid of w and w', the other consisting entirely of even substitutions and generated by the aid of $a_1^5a_2^5 . w$ and $a_1^2a_2^2 . w'$.

From this head we have, therefore, the eight following groups:

$\{H_{243}, s\}$ of order 1215,
$\{H_{243}, s, t\}$ of order 2430_2,
$\{H_{243}, s, a_1^1a_2^1 . a_1^2a_2^2 . a_1^3a_2^3 . a_1^4a_2^4 . a_1^5a_2^5 . t\}$ of order 2430_3,
$\{H_{243}, s, u\}$ of order 4860_2,
$\{H_{243}, s, a_1^1a_2^1 . a_1^2a_2^2 . a_1^3a_2^3 . a_1^4a_2^4 . a_1^5a_2^5 . u\}$ of order 4860_3,
$\{H_{243}, v, v'\}$ of order 14580,
$\{H_{243}, w, w'\}$ of order 29160_2,
$\{H_{243}, a_1^5a_2^5 . w, a_1^2a_2^2 . w'\}$ of order 29160_3.

V. H_6 furnishes one group corresponding to each transitive group of degree 5. These groups are generated respectively by the substitutions s, t, u, v, v', w, w', and can readily be seen to be indentical with those of orders 30_2, 60_2, 120, 360_2, 720 included among the groups with three systems of imprimitivity. An interchange of suffixes and indices in the one set of groups gives the generating substitution of the other set of groups.

VI. H_3 furnishes groups corresponding to the transitive groups of degree 5 by means of the substitutions s, t, u, v, v', w, w'. As in the last case, however, these correspond to the groups of orders 15, 30_1, 60_1, 180, 360_1 included in the groups with three systems of imprimitivity. By the use of the cofactor $\sigma = a_1^1a_2^1 .$

$a_1^2a_2^2 \,.\, a_1^3a_2^3 \,.\, a_1^4a_2^4 \,.\, a_1^5a_2^5$ three more groups can be found generated respectively by the help of $t_1 = \sigma t$, $u_1 = \sigma u$, $w_1 = \sigma w$, $w_1' = \sigma w'$. These groups, however, are seen to be identical with those of orders 30_3, 60_3, and 360_3 included in the groups with three systems of imprimitivity. This head gives, therefore, no group essentially new.

VII. In the discussion of the head unity a useful theorem is the following given by Frobenius (Crelle t CI, p. 287):

The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group, and α the number of its transitive constituents.

The only transitive groups of degree 5 containing 15 as a factor of the order are the symmetric and alternating groups. We have therefore to find an imprimitive group of degree 15 with 5 systems of intransitivity simply isomorphic to the alternating (symmetric) group in 5 letters.

In determining the imprimitive group corresponding to $(A_1A_2A_3A_4A_5)$ pos- we make use of the following facts: (1) the 15 conjugate substitutions corres, ponding to terms of the type $A_1A_2 \,.\, A_3A_4$ must be of degrees 12 or 14; (2) the 20 conjugate substitutions corresponding to terms of the type $A_1\,A_2\,A_3$ must be of degrees 9, 12, or 15; (3) the 24 conjugate substitutions corresponding to terms of the type $A_1A_2A_3A_4A_5$ must be of degree 15. It must, therefore, be possible to solve the equation

$$15\,(12 + 2\alpha) + 20\,(9 + 3\beta) + 24.15 = 14.60$$

where $\alpha = 0, 1$; $\beta = 0, 1, 2$. The only solution is $\alpha = 0$, $\beta = 2$.

Therefore the imprimitive group we are seeking contains among its substitutions 15 of degree 12 and order 2, 20 of degree 15 and order 3, 24 of degree 15 and order 5. Making use of the relations among the generating substitutions of such a group of order 60 as given in Burnside, Theory of Groups, p. 107, we find that the two substitutions corresponding to $A_1A_2A_3A_4A_5$, $A_1A_2A_3A_4$, substitutions which will generate $(A_1A_2A_3A_4A_5)$ pos, are respectively,

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 \,.\, a_2^1a_2^2a_2^3a_2^4a_2^5 \,.\, a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$\rho = a_1^1a_3^2 \,.\, a_1^3a_2^4 \,.\, a_2^1a_1^2 \,.\, a_2^3a_3^4 \,.\, a_3^1a_2^2 \,.\, a_3^3a_1^4;$$

s and ρ are therefore the generating substitutions of an imprimitive group simply isomorphic to the alternating group of degree 5.

In determining a group simply isomorphic to $(A_1A_2A_3A_4A_5)$ all, we argue as before in regard to the various sets of conjugate substitutions. The 15 substitu-

tions corresponding to terms of the type $A_1A_2 \,.\, A_3A_4$ are of degrees 12 or 14, the 20 corresponding to the type $A_1A_2A_3$ are of degrees 9, 12, or 15, the 24 corresponding to the type $A_1A_2A_3A_4A_5$ are of degree 15, the 10 corresponding to the type A_1A_2 are of degrees 6, 8, 10, or 12; the 30 corresponding to the type A_1A_2 A_3A_4 are of degrees 12 or 14; the 20 corresponding to the type $A_1A_2A_3A_4A_5$ are of degree 15. The equation to be satisfied is therefore

$$15\,(12 + 2\alpha) + 20\,(9 + 3\beta) + 24.15 + 10\,(6 + 2\gamma) + 30\,(12 + 2\delta) + 20.15$$
$$= 14.120 \text{ where } \alpha = 0,\ 1\,;\ \beta = 0,\ 1,\ 2;\ \gamma = 0,\ 1,\ 2,\ 3\,;\ \delta = 0,\ 1.$$

The only solution is $\alpha = 0$, $\beta = 2$, $\gamma = 3$, $\delta = 1$. The substitutions $A_1A_2A_3A_4A_5$, $A_2A_3A_4A_5$, $A_1A_2A_3A_4$ will generate the group $(A_1A_2A_3A_4A_5)$ all, and corresponding to these as generators of the imprimitive group we have the three substitutions,

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 \,.\, a_2^1a_2^2a_2^3a_2^4a_2^5 \,.\, a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$\sigma = a_2^1a_3^1 \,.\, a_1^2a_1^3a_1^5a_1^4 \,.\, a_2^2a_3^2a_2^5a_3^4 \,.\, a_3^2a_2^3a_3^5a_2^4,$$
$$\rho = a_1^1a_3^2 \,.\, a_1^3a_2^4 \,.\, a_2^1a_1^2 \,.\, a_2^3a_3^4 \,.\, a_3^1a_2^2 \,.\, a_3^3a_1^4.$$

To sum up the results of the preceding work, the 16 heads with three systems of intransitivity give 41 groups with three systems of imprimitivity. The 7 heads with five systems of intransitivity give 42 groups with five systems of imprimitivity, but of these 13 groups contain also three systems of imprimitivity. Therefore there are 70 imprimitive groups of degree 15 as determined in this paper.

Primitive Substitution Groups of Degree Eighteen.

The main theorems employed in this investigation of primitive groups are the following, in which p is always to stand for a prime number.

I. *The order of a primitive group of degree n cannot exceed* $\dfrac{n!}{2.\,3\,\ldots\,p}$, *where* 2, 3, ... *p are the distinct primes which are less than* $\frac{2}{3}$ *n.* (Burnside, Theory of Groups, p. 199).

II. *A group of degree* $p + x$ *or of degree* $2p + x$, $x > 2$, *cannot be more than x times transitive.* (Miller, Bull. A. M. S., v. IV, pp. 142, 143).

III. *If a primitive group of degree n contains a circular substitution of prime*

order p, the group is at least $(n-p+1)$-fold transitive. (Cole's tr. of Netto's Theory of Substitutions, p. 93).

IV. *A self-conjugate subgroup of a primitive group must be transitive.* (Burnside, l. c., p. 187).

V. *A self-conjugate subgroup of a $\varkappa$-ply transitive group of degree n $(2 < \varkappa < n)$ is in general at least $(\varkappa - 1)$-ply transitive. The only exception is that a triply transitive group of degree 2^m may have a self-conjugate subgroup of order 2^m.* (Burnside, l. c., p. 189).

VI. *A group G which is at least doubly transitive either must be simple or mus contain a simple group H as a self-conjugate subgroup. In the latter case no opera tion of G except identity is permutable with every operation of H. The only exceptions to this statement are that a triply transitive group of degree 2^m may have a self conjugate subgroup of order 2^m, and that a doubly transitive group of degree p^m may have a self-conjugate subgroup of order p^m.* (Burnside, l. c., p. 192).

VII. *The substitutions of a transitive group G which leave a given symbol unchanged form a maximal subgroup G_1, which is one of a set of n conjugate subgroups, each leaving one of the n elements unaffected.* (Burnside, l. c., p. 140).

VIII. *The number of substitutions of degree $l < n$ contained in a transitive group of degree n is equal to the number of substitutions of this same degree l contained in the maximal subgroup G_1 of degree $n-1$ multiplied by $\frac{n}{n-l}$.* (Stated by Miller, Quar. Jour. of. Math. v. XXVIII, p. 215.)

IX. *The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group and α the number of its transitive constituents.* (Frobenius, Crelle, t. CI, p. 287.)

X. Sylow's theorem, as stated by Burnside, l. c., p. 92, or by Sylow, "Théorèmes sur les groupes de substitutions," Math. Ann., v. V (1872), pp. 584 et seq.

XI. *The class of a primitive group of degree n is the same as the class of its maximal subgroup that leaves one element unaffected.*

While the preceding theorems are used throughout the work on primitive groups, the following are used mainly in the determination of simply transitive primitive groups.

XII. *A simply transitive primitive group G of degree n cannot contain a transitive subgroup of degree less than n.* (Miller, Quar. Jour. of Math., v. XXVIII, p. 215.

XIII. *When G_1 contains a self-conjugate subgroup H of degree $n-\alpha$, H must be intransitive, and it must be the transform with respect to substitutions of G of any one of $\alpha - 1$ other subgroups of G_1 $(H'_1, H'_2, \ldots . H'_{\alpha-1})$.* (Miller, Proc. Lon. Math. Soc., v. XXVIII, p. 534.)

XIV. *All the prime numbers which divide the order of one of the transitive constituents of G_1 divide also the orders of each of the other transitive constituents.*

Corollary I. *If one of the transitive constituents of G_1 is of a prime degree, each of its other transitive constituents is of the same or a larger degree, and the order of G_1 is the same as the order of the group formed by these other transitive constituents.*

Corollary II. *If the order of G_1 is not divisible by the square of a prime number, all its transitive constituents are of the same order, and G_1 is formed by establishing a simple isomorphism between them.* (Miller, l. c., p. 536.)

XV. *If a transitive constituent of G_1 is of a prime order, the order of G_1 is the same prime number, and G is of class $n-1$.*

Corollary. *If G_1 contains a constituent of degree 2, its order is 2, and the degree of G is a prime number.* (Miller, l. c., p. 536.)

The above theorems are given in the form and with the symbols most convenient for use, and so are not always exact quotations from the papers and books referred to, while the references given are not always references to the original paper in which the theorem appeared.

Applying these theorems now to the special case in which $n = 18$, we proceed as follows:

Since $18 = 2.7 + 4$, by Theorem II a primitive group cannot be more than 4-ply transitive.

By Theorem I the order is seen not to exceed

$$\frac{18\,!}{2.3.5.7.11} = 2^{15}.\,3^{7}.\,5^{2}.\,7.13.17.$$

If the group included circular substitutions of orders 2, 3, 5, 7, 11, 13, it would be at least 17, 16, 14, 12, 8, 6-fold transitive respectively according to Theorem III. This is impossible; therefore circular substitutions of these orders are not present, and consequently we see at once that 11 and 13 cannot be factors of the order.

If the order includes the factor 7, then, by Theorem X, there is a subgroup of order 7. This must consist of the powers of a substitution composed of two cycles of 7 elements each, and it must be contained self-conjugately in a group of order $7\,.\,4m$ that interchanges transitively among themselves the four remaining elements. (Cf. Burnside, Theory of Groups, p. 202.) It is quite possible to establish a $(7\alpha,\ 1)$ isomorphism between an imprimitive group of degree 14 with the systems of imprimitivity 7, 7 and a transitive group of degree 4; therefore 7 may be a factor of the order.

A subgroup of order 5^2 cannot be present, as it would have to be intransitive with the systems of intransitivity 5, 5 or 5, 5, 5. In the one case, it would have to be contained self-conjugately in a group of order $5^2.\,8m$, in the other, in a group of order $5^2.\,3\,.\,m$. In either case, a circular substitution of order 5 would be present, which is impossible.

The factor 5 may be contained in the order, as it is possible to establish a (5, 1) isomorphism between the cyclical group of degree 15 and the cyclical group in the remaining three letters.

The order must, therefore, be a factor of $2^{15}.\,3^{7}.\,5\,.\,7\,.\,17$.

Simply Transitive Groups.

The maximal subgroup G_1 that leaves a_1 unaffected is intransitive (Theorem XII), and its order is, therefore, a factor of $2^{14}.\,3^{5}.\,5\,.\,7$. Moreover, its class cannot be less than 6, for if it were 2, 3 or 5, G_1 would necessarily contain a transitive subgroup of too low a degree, and it cannot be of class 4, if G is to be primitive. (Netto, l. c., p. 138.)

By Theorem XIV, Cor. I, it is evident that G_1 cannot contain a transitive constituent of degree 13 or 11; by Theorem XIV it cannot contain a transitive

constituent of degree 15 or 14, and by Theorem XV, Cor., it cannot contain a constituent of degree 2.

If one of the transitive constituents of G_1 is of degree 12, the other must be of degree 5. The isomorphism between the transitive groups of degrees 12 and 5 must be an $(\alpha, 1)$ isomorphism, where α itself may be equal to 1. By Theorem XIV, the order of the group of degree 5 must contain the factors 5, 3, 2; therefore this group must be either the alternating or the symmetric group of degree 5. If the isomorphism is more than simple, then the group of degree 12 must be an imprimitive group with 6 systems of imprimitivity. The head for such an imprimitive group as we require is the intransitive group of order 2 and degree 12 given by $[1,\ a_1a_2 \,.\, a_3a_4 \,.\, a_5a_6 \,.\, a_7a_8 \,.\, a_9a_{10} \,.\, a_{11}a_{12}]$. The group $(a_{13}a_{14}a_{15}a_{16}a_{17})$ pos contains

24 conjugate substitutions of order 5 and degree 5,
20 conjugate substitutions of order 3 and degree 3,
15 conjugate substitutions of order 2 and degree 4.

Corresponding to these in the group of degree 17, we have 1 substitution of degree 12 and order 2, 24 of degree $15 + 2\alpha$ and order 5 or 10, 24 others of degree $15 + 2\alpha'$ and order 5 or 10, 20 of degree $15 + 2\beta$ and order 3 or 6, together with 20 of degree $15 + 2\beta'$ and order 3 or 6, 15 of degree $12 + 2\gamma$ and order 2 or 4, and 15 of degree $12 + 2\gamma'$ and order 2 or 4, where

$$\alpha = 0,\ 1;\ \alpha' = 0,\ 1;\ \beta = 0,\ 1;\ \beta' = 0,\ 1;\ \gamma = 0,\ 1,\ 2;\ \gamma' = 0,\ 1,\ 2.$$

By Theorem IX, the following equation must be satisfied:

$$\begin{aligned} 12 + 24\,(15 + 2\alpha) + 24\,(15 + 2\alpha') + 20\,(15 + 2\beta) + 20\,(15 + 2\beta') \\ + 15\,(12 + 2\gamma) + 15\,(12 + 2\gamma') = 120 \,.\, 15. \end{aligned}$$

The only type of solution is given by $\alpha = \beta = \gamma = \beta' = 0$, $\alpha' = 1$, $\gamma' = 2$. G_1, therefore, contains both a self-conjugate subgroup of degree 12 and order 2, and 15 conjugate subgroups of the same type. But, by Theorem XIII, only 5 such conjugate subgroups should exist if this group is to be the G_1 of a simply transitive primitive group. This intransitive group gives us, therefore, no such group as we require.

For precisely the same reason the intransitive group formed by establishing a (2, 1) isomorphism between an imprimitive group of degree 12 and order 240,

and the symmetric group of degree 5 cannot be employed in the formation of a simply transitive primitive group of degree 18.

If the isomorphism is simple, the group G_1, including the alternating group of degree 5, must contain 24 substitutions of degree 10 or 15 and order 5, 20 of degree 6, 9, 12 or 15 and order 3, 15 of degree 6, 8, 10, 12, 14 or 16 and order 2. The following equation must, therefore, be satisfied: $24(10+\alpha) + 20(6+\beta) + 15(6+\gamma) = 60 \times 15$, where $\alpha = 0, 5$; $\beta = 0, 3, 6, 9$; $\gamma = 0, 2, 4, 6, 8, 10$. The only solution is $\alpha = 5$, $\beta = 9$, $\gamma = 10$. G_1 is, therefore, of class 15.

A group G of degree 18, formed with the help of this G_1 and, therefore, of order $60 \cdot 18$, contains 36 conjugate subgroups of order 5, each of which is contained self-conjugately in a group of order 30. As each of these subgroups of order 5 is already self-conjugate in a group of order 10, the construction of the generating substitutions of such a group is an easy matter. G_1 is generated by

$$s = a_1a_3a_7a_5a_9 \cdot a_2a_4a_8a_6a_{10} \cdot a_{13}a_{14}a_{15}a_{16}a_{17}$$

and by

$$t = a_1a_3 \cdot a_2a_4 \cdot a_5a_{12} \cdot a_7a_8 \cdot a_9a_{10} \cdot a_6a_{11} \cdot a_{13}a_{15} \cdot a_{14}a_{17},$$

and contains, as one of the above-mentioned groups of order 10, the group generated by

$$u = st \quad = a_3a_8a_{11}a_6a_9 \cdot a_4a_7a_{12}a_5a_{10} \cdot a_{13}a_{17}a_{15}a_{16}a_{14},$$
$$v = s^{-1}ts = a_1a_2 \cdot a_3a_7 \cdot a_4a_8 \cdot a_5a_6 \cdot a_9a_{12} \cdot a_{10}a_{11} \cdot a_{13}a_{15} \cdot a_{14}a_{16}.$$

The group $\{u, v\}$ is a subgroup of a group of degree 18 and order 30 formed by establishing a (5, 1) isomorphism between an imprimitive group of degree 15 and order 30 with u and its powers as head and the symmetric group in the three elements $a_1a_2a_{18}$. The question then reduces to that of the determination of a substitution of degree 18 and order 3 that will transform the head $\{u\}$ into itself, interchange cyclically the three systems of $\{u\}$, and be in its turn transformed into its square by v. An examination of all substitutions fulfilling these conditions results in finding none that do not give, when combined with other substitutions of G_1, substitutions that cannot possibly belong to a simply transitive primitive group containing G_1 as a maximal subgroup.

There is no primitive or imprimitive group of degree 12 simply isomorphic to the group $(abcdef)_{120}$; consequently, no isomorphism can be established between the symmetric group of degree 5 and a transitive group of degree 12.

There remains the question whether the symmetric group of degree 5 can be put in a simply isomorphic relation to one of the imprimitive groups of degree

12 and order 120 that have both six and two systems of imprivitivity. Such groups of degree 12, however, contain two self-conjugate subgroups of orders 2 and 60 respectively, and, therefore, are not in a simply isomorphic relation to the symmetric group of degree 5.

If one of the transitive constituents is of degree 10, the other can only be of degree 7. By Theorem XIV, the group of degree 10 must contain 7 as a factor of its order, and, therefore, must be either the alternating or the symmetric group. It is impossible to establish an isomorphic relation between either of these groups and one of degree 7 without introducing substitutions of too low a degree.

If one of the transitive constituents is of degree 9, the remaining constituent may be either intransitive in two systems of four elements each or intransitive in eight elements. The isomorphism can in neither case be simple, as an examination of all groups of degree 8 and orders equal to those of transitive groups of degree 9 shows that in each case a system of intransitivity of degree 2 enters, with the single exception of a group of order 144. Here, however, the group of degree 9 contains a substitution of order 8, while an inspection of the corresponding groups of degree 8 shows no substitution of that order.

The isomorphism is, therefore, an (α, β) isomorphism, where α and β are not simultaneously equal to one.

When neither α nor β is equal to one, G_1 must be formed from an imprimitive group of degree 9 and an intransitive group of degree 8. The order of each transitive constituent must contain 3 as a factor, and, therefore, the group of degree 8 must be some combination of the alternating and symmetric groups of degree 4 in two systems of elements. The only combinations possible, consistent with the requirements of class, are got by establishing a simple isomorphism between the two symmetric groups of degree 4 or between the two alternating groups of the same degree. Every relation of isomorphism established between these groups of degree 8 and any imprimitive groups of degree 9 consistent with the requirements of class, results in a G_1 that contains a self-conjugate subgroup of order 4 and degree 8, and no other subgroups of the same order. This case, therefore, gives no simply transitive primitive group.

When α becomes 1, the group of degree 8 must, as before, be composed of either the symmetric or the alternating groups of degree 4 in two sets of elements put into the relation of simple isomorphism. The order of such a group does not contain 9 as a factor; therefore this case gives no possible G_1.

When β becomes 1, the group of degree 9 must be imprimitive. No transitive group of degree 8 stands, however, in the given relation of isomorphism towards an imprimitive group of degree 9. The only permissible intransitive groups of degree 8 are combinations of the symmetric and alternating groups of degree 4 in two sets of elements, and none of these are isomorphic in the given way to any imprimitive group of degree 9.

If one of the transitive constituents is of degree 8, we may have the systems 8, 6, 3 or 8, 3, 3, 3. In both cases we have an $(\alpha, 1)$ isomorphism between an intransitive group of degree 14 and the symmetric group of degree 3. The group of degree 8 is not primitive, as no suitable isomorphic relation can be established between a primitive group of degree 8 and an imprimitive or an intransitive group of degree 6. The only imprimitive groups of degree 8 that can be used are those with the head $(1, a_1a_2 . a_3a_4 . a_5a_6 . a_7a_8)$ that are isomorphic to a group of degree 4 and order 12 or 24. Such groups, however, cannot be combined with the groups in the remaining 9 elements in such a way as to generate a group capable of being the G_1 of one of the required primitive groups.

The case in which G_1 contains a transitive constituent of degree 7 has already been discussed, as according to Theorem XIV, Cor. I, the remaining constituents must be of larger degree.

If G_1 contains a transitive constituent of degree 6, the systems may be either 6, 6, 5 or 6, 4, 4, 3. For the former system the only possible arrangement is to establish a simple isomorphism between the three groups $(a_1a_2a_3a_4a_5a_6)_{60}$, $(a_7a_8a_9a_{10}a_{11}a_{12})_{60}$, $(a_{13}a_{14}a_{15}a_{16}a_{17})$ pos, or between the groups $(a_1a_2a_3a_4a_5a_6)_{120}$, $(a_7a_8a_9a_{10}a_{11}a_{12})_{120}$, $(a_{13}a_{14}a_{15}a_{16}a_{17})$ all. An examination of the two groups G_1 formed from these isomorphisms shows that these are not the maximal subgroups of simply transitive primitive groups of degree 18.

If the systems are 6, 4, 4, 3, only the imprimitive groups of degree 6, the alternating and symmetric groups of degree 4, and the symmetric groups of degree 3 are involved. The group formed by the system 6, 4, 4 has an $(\alpha, 1)$ isomorphism to the group of degree 3, and this isomorphism cannot be simple. No combination of these groups can be found fulfilling all the necessary conditions.

The case in which G_1 contains a transitive system of degree 5 has already been discussed, as the remaining systems must be of degree greater than 5.

If G_1 contains a transitive system of degree 4, the only arrangement possible is 4, 4, 3, 3, 3. The groups involved are therefore the symmetric groups of degree 3 and 4, and the alternating group of degree 4. One group consistent with the

requirements of class is got by establishing a (1, 4) isomorphism between the group,

$$(a_9a_{10}a_{11}\,.\,a_{12}a_{13}a_{14}\,.\,a_{15}a_{16}a_{17})\text{ all, and }(a_1a_2a_3a_4\,.\,a_5a_6a_7a_8)\text{ all.}$$

This group contains, however, one and only one subgroup of degree 8 and order 2.

A second group is got by first establishing a (4, 1) isomorphism between $(a_1a_2a_3a_4.a_5a_6a_7a_8)$ all and $(a_9a_{10}a_{11})$ all; and then establishing a (12, 3) isomorphism between the group of order 24 so formed and the group $(a_{12}a_{13}a_{14}\,.\,a_{15}a_{16}a_{17})$ all. This group of degree 17 contains only one subgroup of order 4 and degree 8; therefore it cannot become a G_1.

G_1 cannot contain only systems of degree less than four, as in such a case a system of degree 2 would have to enter.

There is, therefore, no simply transitive primitive group of degree 18. This result when joined to all other determinations of similar groups shows that there is no simply transitive primitive group of degree $p+1$, p a prime number and $\leqq 17$.

Multiply transitive groups.

Among the transitive groups of degree 17 five contain a self-conjugate subgroup of order 17. These are of order 17, 2.17, 4.17, 8.17, 16.17 respectively, while all excepting the first are of class 16.

If a primitive group of degree 18 and order 18.17 existed, such a group would contain 18 conjugate subgroups of degree 17. It would therefore contain 17 substitutions of degree 18 and 18.16 of degree 17. By Sylow's theorem since $18.17 = 2.3^2.17$, such a group contains either 1 or 34 subgroups of order 3^2. A subgroup of this order must be intransitive, therefore cannot be self conjugate, and it is impossible to form 34 subgroups of order 9 from 17 substitutions of degree 18. No such group of degree 18 exists.

A primitive group of degree 18 and order $18.17.2 = 2^2.3^2.17$ would contain among its substitutions 153 of class 16 and order 2, 288 of class 17 and order 17, 170 of class 18. This group must contain either 1, 4, or 34 conjugate subgroups of order 3^2. As before, a subgroup of this order cannot be self-conjugate, as it is intransitive. If there were 4 conjugate subgroups, each would be self-conjugate in a group of order $3^2.17$ involving all 18 letters and necessarily transitive. Such a group is non-existent. If there were 34 conjugate subgroups they must be of degree 18, and there are not enough substitutions of class 18 to form all these subgroups.

A primitive group of degree 18 and order $18.17.4 = 2^3.3^2.17$ contains among its substitutions 476 of degree 18, 459 of degree 16, 288 of degree 17. According to Sylow's theorem it contains either 1, 3, 9, 17, 51, or 153 conjugate subgroups of order 2^3. Now the group leaving one element unchanged contains 17 conjugate subgroups of degree 16 and order 4; therefore the group of degree 18 contains 153 distinct conjugate subgroups of order 4; therefore it contains 153 conjugate subgroups of order 2^3. Each of these is contained self-conjugately in no larger group.

The number of systems of intransitivity in any one is got from the following equation, where x denotes the number of substitutions of degree 18 and α the number of systems:

$$18x + 16\,(7 - x) = 8\,(18 - \alpha), \text{ where } \alpha \neq 1,\ x < 8.$$

There are two sets of solutions, either $x = 0$, $\alpha = 4$, or $x = 4$, $\alpha = 3$.

The group of degree 17 is generated by,

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$
$$t = a_2a_{14}a_{17}a_5 \,.\, a_3a_{10}a_{16}a_9 \,.\, a_4a_6a_{15}a_{13} \,.\, a_7a_{11}a_{12}a_8\,,$$

where t and its powers form a self-conjugate subgroup of the group of order 2^3 and degree 18 that is now under discussion. It is impossible to so connect the systems and introduce the remaining elements that the first solution may give the group of order 2^3. Making use of the second solution we have only to combine with the group generated by t a substitution of degree 18 that connects the two remaining elements by a transposition, and unites the cycles of t in pairs. The 153 groups of order 2^3 give in this way $153.4 = 612$ distinct substitutions of degree 18, while there are only 476 in the group. This group of degree 18 does not exist.

If there is a primitive group of order $18\,.\,17\,.\,2^3 = 2^4.\ 3^2.\ 17$, it contains 288 substitutions of degree 17, 1071 of degree 16, 1088 of degree 18. The group of degree 17 which is generated by

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$

and $$t = a_2a_{10}a_{14}a_{16}a_{17}a_9a_5a_3 \,.\, a_4a_{11}a_6a_{12}a_{15}a_8a_{13}a_7\,,$$

contains 17 conjugate subgroups of degree 16 and order 8; therefore in the group of degree 18 there are 153 such conjugate subgroups, and each of these is self-conjugate in a group of order 2^4 and degree 18. Denoting by α the number of systems of intransitivity of this group of order 2^4, and letting x denote the number of substitutions of degree 18 contained in the group, we have the equation $18x + 16\,(15 - x) = 16\,(18 - \alpha)$, where $\alpha \neq 1$. There are two solutions, $x = 8$, $\alpha = 2$; $x = 0$, $\alpha = 3$. The first solution would involve a larger number of substi-

tutions of degree 18 than are actually present in the group under consideration. The second solution shows that the group must contain 153 conjugate subgroups of order 2^4 and degree 18 consisting of substitutions of class 16 only, and involving three systems of intransitivity. A substitution must therefore be combined with t that transforms t into one of its powers, and has its head in the group generated by t; moreover, this substitution must have as one of its cycles the transposition $(a_1 a_{18})$, and must have systems of intransitivity apart from this cycle consistent with the systems of t. Such a substitution is $\sigma = a_2a_{10} . a_3 a_{14} . a_9a_{17} . a_5a_{16} . a_6a_{13} . a_7a_{11} . a_8a_{12} . a_1a_{18}$. The required group is therefore $\{s, t, \sigma\}$.

It is not necessary to prove that these three substitutions give a group of the required order, as such a group would be necessarily doubly transitive, and it is known that there is a doubly transitive group of degree 18 and of the required order. By the mode of construction of the substitutions, it is evident that there is only the one type of group of this degree and order.

Any primitive group of degree 18 and order 18 . 17 . 16 contains 2312 substitutions of degree 18, 288 of degree 17, 2295 of degree 16. The group of degree 17 contains 17 conjugate cyclical subgroups of degree 16 and order 16, therefore, the group of degree 18 contains 153 subgroups of order 16, each of which is self-conjugate in one of 153 conjugate subgroups of order 32. Giving α and x the usual meanings, we find that the group of order 32 involves the equation $18x + 16(31 - x) = 32(18 - \alpha)$, where $\alpha \neq 1$. The only solution is $\alpha = 2$, $x = 8$; therefore, the group of degree 12 and order 32 must be intransitive with two systems of intransitivity, and must contain 8 substitutions of degree 18, 23 of degree 16. We have to add, therefore, to the cyclical group of degree 16, 8 substitutions of degree 16 and 8 of degree 18, all of them containing as one cycle the transposition of the remaining two letters.

The group of degree 17 and order 17 . 16 has as generators

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$

and

$$u = a_2a_4a_{10}a_{11}a_{14}a_6a_{16}a_{12}a_{17}a_{15}a_9a_8a_5a_{13}a_3a_7 .$$

The substitution $\tau = a_1a_{18} . a_4a_7 . a_3a_{10} . a_5a_{14} . a_6a_8 . a_9a_{16} . a_{11}a_{13} . a_{12}a_{15}$ generates with s and u the required group of degree 18 and order 17 . 16 . 18. A triply transitive group of such an order is known to exist (Burnside, l. c., p. 158); so no further proof that $\{s, u, \tau\}$ is a group is necessary. It is easy to see that the even substitutions of the group just found form the simple group of order 18 . 17 . 8.

The three remaining transitive groups of degree 17 each contains 120 conjugate subgroups of order 17. They are of orders $15.16.17$, $15.16.17.2$, $15.16.17.4$ respectively.

The group of degree 18 and order $15.16.17.18$ would necessarily contain 816 conjugate subgroups of order 5. Each is self-conjugate in a group of order 90 connecting the remaining three elements transitively. This group is intransitive with two transitive constituents, one of degree 15 and order 90, the other of degree 3. The first, however, is non-existent, therefore, the group of degree 18 is non-existent.

The two remaining groups also, if they can generate primitive groups of degree 18, would generate groups that each contain 816 conjugate subgroups of order 5. In the one case, we should have to make use of an intransitive group containing as a transitive constituent a group of degree 15 and order 180, in the other, the transitive constituent would enter as a group of degree 15 and order 360. Both of these groups are non-existent; therefore, the three groups of degree 17, at present under discussion, furnish us with no new groups of degree 18.

As the case now stands, the conclusion arrived at may be summed up as follows:

There are no simply transitive primitive groups of degree 18, and in addition to the symmetric and alternating groups, there are only two multiply transitive groups of this degree, viz., the two given by

$$\left.\begin{cases} (a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}), \\ (a_2a_{10}a_{14}a_{16}a_{17}a_9a_5a_3 \,.\, a_4a_{11}a_6a_{12}a_{15}a_8a_{13}a_7), \\ (a_2a_{10} \,.\, a_3a_{14} \,.\, a_9a_{17} \,.\, a_5a_{16} \,.\, a_6a_{13} \,.\, a_7a_{11} \,.\, a_8a_{12} \,.\, a_1a_{18}), \end{cases}\right\} \text{of order } 2448,$$

$$\left.\begin{cases} (a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}), \\ (a_2a_4a_{10}a_{11}a_{14}a_6a_{16}a_{12}a_{17}a_{15}a_9a_8a_5a_{13}a_3a_7), \\ (a_1a_{18} \,.\, a_4a_7 \,.\, a_3a_{10} \,.\, a_5a_{14} \,.\, a_6a_8 \,.\, a_9a_{16} \,.\, a_{11}a_{13} \,.\, a_{12}a_{15}), \end{cases}\right\} \text{of order } 4896.$$

The second of these is triply transitive, and contains the first, which is doubly transitive and simple, as a self-conjugate subgroup.

The works consulted in the preparation of this paper have included, in addition to the standard works on the subject by Jordan, Serret, Netto, and Burnside, the following papers:

Askwith, "On Possible Groups of Substitutions that can be formed with 3, 4, 5, 6, 7 Letters Respectively." Quar. Jour. Math., v. XXIV (1890), pp. 11–167.

"On Groups of Substitutions that can be formed with Eight Letters." Quar. Jour. Math., v. XXIV (1890), pp. 263–331.

"On Groups of Substitutions that can be formed with Nine Letters." Quar. Jour. Math., v. XXVI (1892), pp. 79–128.

Cayley, "On Substitution Groups for Two, Three, Four, Five, Six, Seven, and Eight Letters." Quar. Jour. Math., v. XXV (1891), pp. 71–88, 137–155.

Cole, "List of the Substitution Groups of Nine Letters." Quar. Jour. Math., v. XXVI (1892), pp. 372–388.

"The Transitive Substitution Groups of Nine Letters." Bull. New York Math. Soc., v. II (1893), pp. 250–258.

"List of the Transitive Substitution Groups of Ten and of Eleven Letters." Quar. Jour. Math., v. XXVII (1894), pp. 39–50.

"Note on the Substitution Groups of Six, Seven, and Eight Letters." Bull. New York Math. Soc., v. II (1893), pp. 184–190.

Miller, "Intransitive Substitution Groups of Ten Letters." Quar. Jour. Math., v. XXVII (1894), pp. 99–118.

"List of Transitive Substitution Groups of Degree Twelve." Quar. Jour. Math., v. XXVIII (1896), pp. 193–231.

"Note on the Transitive Substitution Groups of Degree Twelve." Bull. Amer. Math. Soc., v. I (1894–1895), pp. 255–258.

"On the Transitive Substitution Groups of Degrees Thirteen and Fourteen." Quar. Jour. Math., v. XXIX (1898), pp. 224–249.

"On the Primitive Substitution Groups of Degree Fifteen." Proc. Lon. Math. Soc., v. XXVIII (1896–1897), pp. 533–544.

"On the Primitive Substitution Groups of Degree Sixteen." Amer. Jour. Math., v. XX (1898), pp. 229–241.

"On the Transitive Substitution Groups of Degree Seventeen." Quar. Jour. Math., v. XXXI (1899), pp. 49–57.

"A Simple Proof a Fundamental Theorem of Substitution Groups, and Several Applications of the Theorem." Bull. Amer. Math. Soc., v. II (1895–1896), pp. 75–77.

"On the Limit of Transitivity of the Multiply Transitive Substitution Groups that do not Contain the Alternating Group." Bull. Amer. Math. Soc., v. IV (1897–1898), pp. 140–143.

Jordan, "Sur la classification des groupes primitifs." Comptes Rendus, t. LXXIII (1871), pp. 853–857.

"Sur l'énumeration des groupes primitifs pour les dix-sept premiers degrés." Comptes Rendus, t. LXXV (1872), pp. 1754–1757.

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Removal of any Two Terms from a Binary Quantic by Linear Transformations.

By Bessie Growe Morrison.

In his "Bericht," p. 179, Meyer has enumerated the various steps in the history of this subject.

More recently, H. B. Newson (Annals of Math., vol. XI, Nos. 3 and 4, 1897) showed that every non-singular quantic of odd degree may be linearly transformed so that its middle term shall vanish.

We propose to discuss those linear transformations by which any two terms of a binary quantic disappear.

§1.—Covariants $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$.

$$\left.\begin{array}{l} r=(1,\,2\,\ldots\ldots n-1) \\ \kappa=(1,\,2\,\ldots\ldots n-1) \end{array}\right) n-r>\kappa.$$

If to the quantic $f=\alpha_z^n$ we apply the transformation

$$T{:}\begin{array}{l} z_1=\lambda_1 z_1'+\mu_1 z_2' \\ z_2=\lambda_2 z_1'+\mu_2 z_2' \end{array}\Bigg\}\,,\ \Delta\neq 0,$$

then f takes the form

$$(A)\ f'(z')=\alpha_\lambda^n z_1'^n+\tbinom{n}{1}\alpha_\lambda^{n-1}\alpha_\mu z_1'^{n-1}z_2'+\ldots\ldots\alpha_\mu^n z_2'^n.$$

If by a transformation T any two terms of f' vanish simultaneously, the necessary and sufficient condition is

$$\begin{array}{ll} (1) & \alpha_\lambda^r\alpha_\mu^{n-r}=0 \\ (2) & \alpha_\lambda^{n-\kappa}\alpha_\mu^\kappa=0 \end{array}\left\{\begin{array}{l} r=(0,1,2,\ldots\ldots n-1) \\ \kappa=(0,1,2,\ldots\ldots n-1) \end{array}\right) n-r>\kappa.$$

The condition that (1) and (2) vanish simultaneously is that the resultant in μ, $R_{n-r,\kappa}$, or the resultant in λ, $R_{r,n-\kappa}$, vanish.

Hence the problem is to eliminate λ or μ between (1) and (2).

Either resultant contains the factor f. Let μ be a root of $f = \alpha_z^n$, then $\alpha_\mu^n = 0$.

$$(1)\quad f_{(\mu)^{n-r}} = \alpha_z^r\, \alpha_\mu^{n-r},$$
$$(2)\quad f_{(\mu)^\kappa} = \alpha_z^{n-\kappa}\, \alpha_\mu^\kappa.$$

It is evident that (1) and (2) vanish simultaneously for $z = \mu$;

since $$\alpha_\mu^r\, \alpha_\mu^{n-r} = \alpha_\mu^n = 0,$$
and $$\alpha_\mu^{n-\kappa}\, \alpha_\mu^\kappa = \alpha_\mu^n = 0.$$

Therefore f is a factor of $R_{n-r,\kappa}$ and similarly if λ is a root of f, then also is f a factor of $R_{r,n-\kappa}$.

Now if λ and μ have values which satisfy $f = \alpha_z^n$, $\alpha_\lambda^n = 0$ and $\alpha_\mu^n = 0$. Hence in A the first and last terms vanish. Since this case is so simple we exclude it by dividing f out of $R_{n-r,\kappa}$ and $R_{r,n-\kappa}$ calling the remaining functions $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ respectively, where

$$\left\{\begin{array}{l} r = (1, 2, \ldots\ldots n-1) \\ \varkappa = (1, 2, \ldots\ldots n-1) \end{array}\right\},\ n - r > \varkappa,$$

$R_{n-r,\kappa}$ and $R_{r,n-\kappa}$ are both of degree $[(n-r)(n-\varkappa) + \varkappa r]$,

and $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ are both of degree $[(n-r)(n-\varkappa) + \varkappa r - n]$

§2.—PROPERTIES OF λ-POINTS AND μ-POINTS.

From the consideration of

$$\left.\begin{array}{ll} (1) & f_{(\mu)^{n-r}} = \alpha_z^r \alpha_\mu^{n-r} \\ (2) & f_{(\mu)^\kappa} \ \ = \alpha_z^{n-\kappa} \alpha_\mu^\kappa \end{array}\right\}$$

and

$$\left.\begin{array}{ll} (3) & f_{(\lambda)^r} \ \ = \alpha_z^{n-r} \alpha_\lambda^r \\ (4) & f_{(\lambda)^{n-\kappa}} = \alpha_z^\kappa \alpha_\lambda^{n-\kappa} \end{array}\right\}$$

it can be shown that the z-points, or common values of (1) and (2), are the λ-points $R_{r,n-\kappa}$, and the z-points, or common values of (3) and (4), are the μ-points of $R_{n-r,\kappa}$; i. e. if the μ's are the roots of $R_{n-r,\kappa}$ and the λ's the roots of $R_{r,n-\kappa}$, then

$$(5)\quad \alpha_\lambda^r\, \alpha_\mu^{n-r} = 0,$$
$$(6)\quad \alpha_\lambda^{n-\kappa}\, \alpha_\mu^\kappa = 0.$$

Similarly if the λ's are the roots of $R_{n-r,\kappa}$ and the μ's the roots of $R_{r,n-\kappa}$ then

$$(7) \quad \alpha_\lambda^{n-r}\alpha_\mu^r = 0.$$
$$(8) \quad \alpha_\lambda^\kappa \alpha_\mu^{n-\kappa} = 0.$$

There is a one-to-one correspondence between the μ's and λ's: For the root μ_1 of the resultant $R_{n-r,\kappa}$ there is a common root λ_1 to (1) and (2); also for the root λ_1 of $R_{r,n-\kappa}$ (3) and (4), have the root μ_1 in common.

Give to μ in T the values of the roots of $C_{n-r,\kappa}$ and to λ the values of corresponding roots of $C_{r,n-\kappa}$, calling the resulting transformations $T_{n-r,\kappa}$, then by $T_{n-r,\kappa}$, the $(n-r+1)^{st}$ and $(\kappa+1)^{st}$ terms of f vanish.

Interchange λ's and μ's, obtaining $T_{r,n-\kappa}$, and by these transformations the $(r+1)^{st}$ and $(n-\kappa+1)^{st}$ terms of f' vanish.

$T_{n-r,\kappa}$ and $T_{r,n-\kappa}$ each consist of $[(n-r)(n-\kappa)+\kappa r-n]$ transformations.

Case $\kappa=r$; $R_{n-r,\kappa}$ and $R_{r,n-\kappa}$ coincide and we call this function $\overline{C_r}$, which is of degree $2r(r-n)+n(n-1)$, an even number.

It is easy to show that the roots of $\overline{C_r}$ go in pairs which have reciprocal relations; the root common to r^{th} and $(n-r)^{th}$ polars of μ_1 is λ_1 and the root common to $(n-r)^{th}$ and r^{th} polars of λ_1 is μ_1.

This reciprocal relation we call correspondence, and denote the corresponding roots by:

$$\mu_p, \lambda_p \left(p = 1, 2, \ldots \frac{2r(r-n)+n(n-1)}{2}\right).$$

Give to λ and μ in T values of pairs of corresponding roots of $\overline{C_r}$ and we get a set of $2r(r-n)+n(n-1)$ transformations $\overline{T_r}$, by which the $(r+1)^{st}$ and $(n-r+1)^{st}$ terms of f' vanish.

§3.—Form of $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$.

1) $r=1$

$$(1) \quad f_{(\mu)^{n-1}} = \alpha_z\alpha_\mu^{n-1},$$
$$(2) \quad f_{(\mu)^\kappa} = \alpha_z^{n-\kappa}\alpha_\mu^\kappa.$$

To compute $R_{n-r,\kappa}$ in symbolic notation put

$$z_1 = \rho\beta_\mu^{n-1}\beta_2,$$
$$z_2 = -\rho\beta_\mu^{n-1}\beta_1,$$

then $$R_{n-1,\kappa} = \alpha_\mu^\kappa \left\{ \begin{matrix} [\beta_\mu^{n-1} \gamma_\mu^{n-1} \delta_\mu^{n-1} \text{ .... to } (n-\varkappa) \text{ factors}]. \\ [(\alpha\beta)(\alpha\gamma)(\alpha\delta) \text{ .... to } (n-\varkappa) \text{ factors}]. \end{matrix} \right\}.$$

2) $r = \varkappa = 1$.

$$\overline{R_1} = \alpha_\mu [\beta_\mu^{n-1} \gamma_\mu^{n-1} \text{ .... to } (n-1) \text{ factors}] \, [(\alpha\beta)(\alpha\gamma) \text{ .... to } (n-1) \text{ factors}].$$

3) $r = 2$, $(n-\varkappa)$ even*

$$R_{n-2,\kappa} = \alpha_\mu^\kappa \alpha_\mu'^\kappa (\alpha\alpha')^{n-\kappa} \left[\beta_\mu^{n-2} \beta_\mu'^{n-2} (\beta\beta')^2 \gamma_\mu^{n-2} \gamma_\mu'^{n-2} (\gamma\gamma')^2 \text{ .... to } \frac{n-\varkappa}{2} \text{ factors}\right].$$

$$-\frac{n-\varkappa}{2} \cdot \alpha_\mu^\kappa \alpha_\mu'^\kappa (\alpha\alpha')^{n-\kappa-2} . \beta_\mu^{n-2} \beta_\mu'^{n-2} (\alpha\beta)^2 (\alpha'\beta')^2 \Big[\gamma_\mu^{n-2} \gamma_\mu'^{n-2}$$
$$\times (\gamma\gamma')^2 \text{ .... to } \frac{n-\varkappa-2}{2} \text{ factors}\Big].$$
$$+ \frac{(n-\varkappa)(n-\varkappa-3)}{2^2 . 2!} \alpha_\mu^\kappa \alpha_\mu'^\kappa (\alpha\alpha')^{n-\kappa-4} . \beta_\mu^{n-2} \beta_\mu'^{n-2} \gamma_\mu^{n-2} \gamma_\mu'^{n-2} (\alpha\beta)(\alpha'\gamma)$$
$$(\alpha\beta')(\alpha'\gamma') . \left[\delta_\mu^{n-2} \delta_\mu'^{n-2} (\delta\delta')^2 \text{ .... to } \frac{n-\varkappa-4}{2} \text{ factors}\right]$$
$$+ \text{ .... etc. } + \text{ ....}$$
$$+ \frac{(-1)^{\frac{n-\kappa}{2}}}{(2)^{\frac{n-\kappa-2}{2}}} \left\{ \alpha_\mu^\kappa \alpha_\mu'^\kappa \begin{bmatrix} . \beta_\mu^{n-2} \beta_\mu'^{n-2} (\alpha\beta)^2 (\alpha\beta')^2 \text{ .... to } \frac{n-\varkappa}{2} \text{ factors} \\ . \gamma_\mu^{\mu-2} \gamma_\mu'^{n-2} (\alpha'\gamma)^2 (\alpha'\gamma')^2 \text{ .... to } \frac{n-\varkappa}{2} \text{ factors} \end{bmatrix} \right\}.$$

4) Means of obtaining $C_{n-r,\kappa}$ or $C_{r,n-\kappa}$ when formulæ become complicated.

a) Let a be a root of $C_{n-r,\kappa}$ and c the corresponding root of $C_{r,n-\kappa}$. In T give μ and λ these values respectively,

then $$z_1 = c_1 z_1' + a_1 z_2',$$
$$z_2 = c_2 z_1' + a_2 z_2',$$

which is a transformation of the set $T_{n-r,\kappa}$.

Apply this transformation to,

$$C_{r,n-\kappa} = (cz)(\text{........}).$$

then $$C'_{r,n-\kappa} = \rho_1 z_2' (\text{........}).$$

Therefore the first coefficient of $C'_{r,n-\kappa}$ must be zero.

---

* See "Resultant of Binary Quadric and n-ic" by Professor H. S. White, Bulletin of A. M. Soc., 2nd series, Vol. I, No. I. Also Clebsch, "Binare Formen," pp. 84–88. In this paper Professor White points out the changes necessary if we assume the order of f as odd, and we find that his formula would become much more complex, hence our formula would be extremely so.

b) Let a and c be two corresponding roots of $\overline{C}_r$,
then

$$\begin{aligned} z_1 &= c_1 z_1' + a_1 z_2' \\ z_2 &= c_2 z_1' + a_2 z_2'. \end{aligned}$$

belongs to $\overline{T}_r$.

Apply this transformation to

$$\overline{C}_r = (az)(cz)(\ldots\ldots\ldots),$$

then

$$\overline{C}_r' = \rho_2 z_1' z_2' \quad (\ldots\ldots\ldots).$$

Therefore the first and last coefficients of $\overline{C}_r$ must be zero. To sum up for future reference:

by $T_{n-r,\kappa}$ the $(n-r+1)^{st}$ and $(\kappa+1)^{st}$ terms of f' vanish,
by $T_{n-r,\kappa}$ the last term of $C_{n-r,\kappa}$ and the first term of $C_{r,n-\kappa}$ vanish,
by $T_{r,n-\kappa}$ the $(r+1)^{st}$ and $(n-\kappa+1)^{st}$ terms of f' vanish,
by $T_{r,n-\kappa}$ the first term of $C_{n-r,\kappa}$ and the last term of $C_{r,n-\kappa}$ vanish.

Using these results in the applications, we can determine the form of the desired covariant in each particular case.

§4.—Applications of the Theory.

1) *The Cubic.*

$n=3$, $n-r>\kappa$, hence the only values for r and κ are $r=\kappa=1$.

$$\begin{aligned} f_{(\mu)^2} &= \alpha_z \alpha_\mu^2, \\ f_{(\mu)} &= \alpha_z^2 \alpha_\mu. \\ \therefore \overline{R}_1 &= \alpha_\mu \beta_\mu^2 \gamma_\mu^2 (\alpha\beta)(\alpha\gamma) \\ &= \tfrac{1}{2} f. (\alpha\beta)^2 \alpha_\mu \beta_\mu. \end{aligned}$$

or

$$\overline{C}_1{}^* = (\alpha\beta)_2 \alpha \ \beta_\mu = \Delta \text{ of Clebsch}.$$

If $\overline{C}_1 = (\alpha\beta)^2 \alpha_\mu \beta_\mu = (z'\xi)(z'\eta)$, we can give to λ and μ the values of the roots of $\overline{C}_1$.

Hence f can be transformed by $\overline{T}_1$ in two ways so that its 2nd and 3rd terms shall vanish.

2) *Quartic.*

there are two cases, $r=\kappa=1$; $r=1$, $\kappa=2$.

---

* Always omitting the numerical factor since we are to consider only the roots of the covariant.

a) $r = \varkappa = 1$.

$$f_{(\mu)^3} = \alpha_z \alpha_\mu^3,$$
$$f_{(\mu)} = \alpha_z^3 \alpha_\mu.$$

$$\overline{R}_1 = \alpha_\mu \beta_\mu^3 \gamma_\mu^3 \delta_\mu^3 (\alpha\beta)(\alpha\gamma)(\alpha\delta),$$
$$= f.\, \alpha_\mu \beta_\mu^2 \delta_\mu^3 (\alpha\beta)^2 (\alpha\delta),$$
$$= f.\, T.$$
$$C_1 = \alpha_\mu \beta_\mu^2 \delta_\mu^3 (\alpha\beta)^2 (\alpha\delta) = T \text{ of Clebsch.}$$

By the six transformations $\overline{T_1}$ the 2nd and 4th terms of f vanish.

b) $r = 1,\ \varkappa = 2$.

$$f_{(\mu)^3} = \alpha_z \alpha_\mu^3,$$
$$f_{(\mu)^2} = \alpha_z^2 \alpha_\mu^2.$$
$$R_{3,2} = \alpha_\mu^2 \beta_\mu^3 \gamma_\mu^3 (\alpha\beta)(\alpha\gamma)$$
$$= \tfrac{1}{2} f (\alpha\gamma)^2 \alpha_\mu^2 \gamma_\mu^2 = \tfrac{1}{2} f.\, H.$$
$$C_{3,2} = (\alpha\gamma)^2 \alpha_\mu^2 \gamma_\mu^2 = H.$$
$$f_{(\lambda)} = \alpha_z^3 \alpha_\lambda,$$
$$f_{(\lambda)^2} = \alpha_z^2 \alpha_\lambda^2.$$

From §3, 4), *a*), if we apply a transformation of the set $T_{n-r,\kappa}$ to $C_{r,n-\kappa}$, its first coefficient will vanish. Also by the transformations $T_{n-r,\kappa}$ the $(n - r, + 1)^{\text{st}}$ and $(\varkappa + 1)^{\text{st}}$ terms of f' vanish.

But by actual trial $(Hi - fj)$ is the only combination of fundamental covariants whose first coefficient in the transformed form vanishes, and which is of the same degree and order as $C_{1,2}$.

Therefore $C_{1,2} = (Hi - fj)$, where i and j are the invariants of the quartic.

The quartic can be linearly transformed by the four transformations $T_{3,2}$ so that its 3rd and 4th terms vanish, and by $T_{1,2}$ so that its 2nd and 3rd terms vanish.

3) *Quintic.*

There are four cases; $r = \varkappa = 1$; $r = \varkappa = 2$; $r = 1,\ \varkappa = 2$; $r = 1,\ \varkappa = 3$.

a) $r = \varkappa = 1$.

$$f_{(\mu)^4} = \alpha_z \alpha_\mu^4,$$
$$f_{(\mu)} = \alpha_z^4 \alpha_\mu,$$
$$\overline{R}_1 = \alpha_\mu \beta_\mu^4 \gamma_\mu^4 \delta_\mu^4 \varepsilon_\mu^4 (\alpha\beta)(\alpha\gamma)(\alpha\delta)(\alpha\varepsilon).$$

To express this in terms of fundamental covariants;* multiply

$$2\beta_\mu \gamma_\mu (\alpha\beta)(\alpha\gamma) = \beta_\mu^2 (\alpha\gamma)^2 + \gamma_\mu^2 (\alpha\beta)^2 - \alpha_\mu^2 (\beta\gamma)^2,$$

by $$2\delta_\mu \varepsilon_\mu (\alpha\delta)(\alpha\varepsilon) = \delta_\mu^2 (\alpha\varepsilon)^2 + \varepsilon_\mu^2 (\alpha\delta)^2 - \alpha_\mu^2 (\delta\varepsilon)^2,$$

and then by $$\alpha_\mu \beta_\mu^3 \gamma_\mu^3 \delta_\mu^3 \varepsilon_\mu^3.$$

Assembling identical terms

$$4\,\overline{R}_1 = 4\alpha_\mu \beta_\mu^3 \gamma_\mu^5 \delta_\mu^3 \varepsilon_\mu^5 (\alpha\beta)^2 (\alpha\delta)^2 - 3f.\,4H^2,$$
$$= 4f^3.\,S - 12f.\,H,$$
$$\overline{C}_1 = f^2 S - 3H^2. \qquad \text{where } \beta_\mu^3 \gamma_\mu^3 (\beta\gamma) = 2H,$$
$$\alpha_\mu \beta_\mu (\alpha\beta)^4 = 2S.$$

$\overline{C}_1$ furnishes twelve transformations by which the 2nd and 5th terms of f' vanish.

b) $r = \varkappa = 2$,.

$$f_{(\mu)^3} = \alpha_z^2 \alpha_\mu^3,$$
$$f_{(\mu)^2} = \alpha_z^3 \alpha_\mu^2,$$

From §3, 4), *b*), by $\overline{T}_2$ the first and last coefficients of $\overline{C}_2'$ vanish; also by $\overline{T}_2$ the 3rd and 4th terms of f' vanish.

Therefore $\overline{C}_1 = (HI - fJ)$, where I is the covariant of the quintic of degree 2 and order 2, and J is the covariant of degree 3 and order 3.

c) $r = 1$, $\varkappa = 2$.

$$f_{(\mu)^4} = \alpha_z \alpha_\mu^4,$$
$$f_{(\mu)^2} = \alpha_z^3 \alpha_\mu^2.$$
$$R_{4,2} = \alpha_\mu^2 \beta_\mu^4 \gamma_\mu^4 \delta_\mu^4 (\alpha\beta)(\alpha\gamma)(\alpha\delta),$$
$$= f\,\alpha_\mu^2 \beta_\mu^3 \delta_\mu^4 (\alpha\beta)^2 (\alpha\delta)$$
$$= f.\,T.$$
$$C_{4,2} = \alpha_\mu^2 \beta_\mu^3 \delta_\mu^4 (\alpha\beta)^2 (\alpha\delta) = T.$$
$$f_{(\lambda)} = \alpha_z^4 \alpha_\lambda,$$
$$f_{(\lambda)^3} = \alpha_z^2 \alpha_\lambda^3.$$

By the transformations $T_{4,2}$ the 3rd and 5th terms of f' vanish and the first coefficient of $C_{1,3}'$.

$$C_{1,3} = (fS^2 - 16HT) \text{ in Salmon's notation.}$$

---

* Salmon, "Mod. Higher Alg.," p. 319.

d) $r = 1, \varkappa = 3$.

$$
\begin{aligned}
f_{(\mu)^4} &= \alpha_z \, \alpha_\mu^4 , \\
f_{(\mu)^3} &= \alpha_z^2 \, \alpha_\mu^3 ; \\
R_{4,3} &= \alpha_\mu^3 \, \beta_\mu^4 \, \gamma_\mu^4 \, (\alpha\beta) \, (\alpha\gamma). \\
&= \tfrac{1}{2} f. \, H. \\
C_{4,3} &= H. \\
f_{(\lambda)} &= \alpha_z^4 \, \alpha_\lambda , \\
f_{(\lambda)^4} &= \alpha_z^3 \, \alpha_\lambda^2 .
\end{aligned}
$$

By $T_{4,3}$ the 4th and 5th terms of f vanish and the first coefficient of $C'_{1,2}$.

Therefore $C_{1,2} = (I^3 - 27 \, J^2)$ where I and J are the same covariants mentioned before for the quintic.

4) *Sextic.*

There are six cases: $r = \varkappa = 1$; $r = \varkappa = 2$; $r = 1$, $\varkappa = 2$; $r = 1$, $\varkappa = 3$; $r = 2$, $\varkappa = 3$; $r = 1$, $\varkappa = 4$.

a) $r = \varkappa = 1$.

$$
\begin{aligned}
f_{(\mu)^5} &= \alpha_\mu \, \alpha_\mu^5 , \\
f_{(\mu)} &= \alpha_\mu^5 \, \alpha_\mu , \\
\overline{R}_1 &= \alpha_\mu \, \beta_\mu^5 \, \gamma_\mu^5 \, \delta_\mu^5 \, \varepsilon_\mu^5 \, i_\mu^5 \, (\alpha\beta) \, (\alpha\gamma) \, (\alpha\delta) \, (\alpha\varepsilon) \, (\alpha i).
\end{aligned}
$$

To express $\overline{R}_1$ in terms of fundamental covariants multiply*

$$(\alpha\beta) \, (\alpha\delta) \, \beta_\mu \, \delta_\mu = \tfrac{1}{2} \left[(\alpha\beta)^2 \, \delta_\mu^2 + (\alpha\delta)^2 \, \beta_\mu^2 - (\beta\delta)^2 \, \alpha_\mu^2 \right],$$

by

$$(\alpha\gamma) \, (\alpha\varepsilon) \, \gamma_\mu \, \varepsilon_\mu = \tfrac{1}{2} \left[(\alpha\gamma)^2 \, \varepsilon_\mu^2 + (\alpha\varepsilon)^2 \, \gamma_\mu^2 - (\gamma\varepsilon)^2 \, \alpha_\mu^2 \right],$$

and then by

$$\alpha_\mu \, \beta_\mu^4 \, \gamma_\mu^4 \, \delta_\mu^4 \, \varepsilon_\mu^4 \, i_\mu^5 \, (\alpha i) ;$$

we get

$$
\begin{aligned}
\overline{R}_1 &= f \left[2 \, HT - f^2 \, S \right], \\
\overline{C}_1 &= 2 \, HT - f^2 \, S.
\end{aligned}
$$

where

$$
\begin{aligned}
T &= -(\alpha\beta)^2 \, (\alpha i) \, \alpha_\mu^3 \, \beta_\mu^4 \, i_\mu^5 , \\
S &= -(\alpha\beta)^2 \, \alpha_\mu \, \beta_\mu^2 \, i_\mu^5 , \\
2 \, H &= \quad (\gamma\varepsilon)^2 \, \gamma_\mu^4 \, \varepsilon_\mu^4 .
\end{aligned}
$$

b) $r = \varkappa = 2$.

$$
\begin{aligned}
f_{(\mu)^4} &= \alpha_\mu^2 \, \alpha_\mu^4 , \\
f_{(\mu)^2} &= \alpha_z^4 \, \alpha_\mu^4 .
\end{aligned}
$$

* Stephanos, "Mém. des Sav. Etr.," Vol. XXVII, p. 116.

Even in this case the formula §3, 3) is complicated and it is not easy to express in fundamental covariants in such a way that f will divide out.

By $\overline{T}_2$ the 3rd and 5th terms of f' vanish and also the first and last coefficients of C_2'.

Therefore $\qquad \overline{C}_2 = (fS^2 - 16\ HT)$.

c) $r = 1,\ \varkappa = 2$.

$$f_{(\mu)^5} = a_z\, a_\mu^5,$$
$$f_{(\mu)^2} = a_z^4\, a_\mu^2.$$
$$R_{5,2} = a_\mu^2\, \beta_\mu^5\, \gamma_\mu^5\, \delta_\mu^5\, \varepsilon_\mu^5\, (a\beta)\,(a\gamma)\,(a\delta)\,(a\varepsilon).$$

By $T_{1,4}$ the 2nd and 5th terms of f' vanish and the 1st term of $C'_{5,2}$.

Therefore $\qquad C_{5,2} = (f^2S - 3\ H^2)$ Salmon's notation.

$$f_{(\lambda)} = a_z^5\, a_\lambda,$$
$$f_{(\lambda)^4} = a_z^2\, a_\lambda^4.$$

By $T_{5,2}$ the 3rd and 6th terms of f' vanish and the first term of $C'_{1,4}$.

Therefore

$$C_{1,4} = C_{1.6}^2 \cdot C_{1.4} - 2\, C_{1.6}^2 \cdot I_2 \cdot C_{2.4} + 36\, C_{1.6} \cdot C_{2.4}\, C_{3.6} + 3\, [C_{3.8}^2 + 4\, C_{2.4}^2 \cdot C_{2.8}].$$

(Notation of E. B. Elliott in "Algebra of Quantics.")

d) $r = 1,\ \varkappa = 3$.

$$f_{(\mu)^5} = a_z\, a_\mu^5,$$
$$f_{(\mu)^3} = a_z^3 a_\mu^3.$$
$$R_{5,3} = a_\mu^3\, \beta_\mu^5\, \gamma_\mu^5\, \delta_\mu^5\, (a\beta)\,(a\gamma)\,(a\delta),$$
$$= f.\ a_\mu^3\, \beta_\mu^4\, \gamma_\mu^5\, (a\beta)^2\,(a\delta),$$
$$= f.\ T.$$
$$C_{5,3} = a_\mu^3\, \beta_\mu^4\, \gamma_\mu^5\, (a\beta)^2\,(a\delta) = T.$$
$$f_{(\lambda)} = a_z^5\, a_\lambda,$$
$$f_{(\lambda)^3} = a_z^3\, a_\lambda^3.$$

$C_{1.3}$ is of degree twelve and order seven, but the covariants required to express it are of too high an order to handle easily.

e) $r = 2,\ \varkappa = 3$.

$$f_{(\mu)^4} = a_z^2\, a_\mu^4,$$
$$f_{(\mu)^3} = a_z^3\, a_\mu^3.$$

By $T_{2,3}$ the 3rd and 4th terms of f' vanish and the first term of $C'_{4,3}$. Therefore

$$C_{4.3} = HS - fT. \quad \text{(Salmon's notation).}$$

f). $r = 1$, $\varkappa = 4$.

$$f_{(\mu)^5} = \alpha_z \alpha_\mu^5,$$
$$f_{(\mu)^4} = \alpha_z^2 \alpha_\mu^4.$$
$$R_{5,4} = f.\ H,$$
$$C_{5,4} = H.$$
$$f_{(\lambda)} = \alpha_z^5 \alpha_\mu,$$
$$f_{(\lambda)^2} = \alpha_z^4 \alpha_\mu^2.$$

In this case $C_{1,2}$ is complicated and requires covariants of a high order to express it.

Whenever the 2nd and 3rd terms of a quantic vanish $C_{n-r,\kappa}$ is always the Hessian, and whenever the 2nd and 4th terms of f vanish $C_{n-r,\kappa}$ is the Jacobian or T.

In general when two adjacent terms of a quantic vanish $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ will be Hessians of some order; and when only one term appears between the two which vanish, then $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ will be Jacobians of some order.

A Hessian of higher order may be defined up to a numerical factor as the result of eliminating z between.

$$\begin{array}{l} \alpha_z^r \alpha_\mu^{n-r} \\ \alpha_z^{r+1} \alpha_\mu^{n-r-1} \end{array} \qquad r = (1, 2, \ldots\ldots n-2)$$

and then dividing out $f = \alpha_\mu^n$.

Similarly a Jacobian of higher order may be defined up to a numerical factor as the result of eliminating z between.

$$\begin{array}{l} \alpha_z^r \alpha_\mu^{n-r} \\ \alpha_z^{r+2} \alpha_\mu^{n-r-2} \end{array} \qquad (r = 1, 2, \ldots\ldots n-3),$$

and then dividing out $f = \alpha_z^n$.

When $r = 1, 2, 3$ .... etc., we have Hessians and Jacobians of the 1st, 2nd, 3rd .... etc., orders respectively.

These examples comprise all the canonical forms, in which two terms vanish simultaneously by linear transformation, of the non-singular cubic, quartic, quintic and sextic.

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Memoir on the Algebra of Symbolic Logic.

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PART II.

The Theory of Substitutions.

§1.—*Types of Transformation.*

Any transformation of x and y into u and v can be represented by

$$x = f_1(u, v), \quad y = f_2(u, v). \tag{1}$$

Here $f_1(u, v)$ and $f_2(u, v)$ will be called the director functions of the transformation. They will always be represented by the following notation:

$$\left.\begin{aligned} f_1(u, v) &= a_1 uv + a_2 u\bar{v} + a_3 \bar{u}v + a_4 \bar{u}\bar{v}, \\ f_2(u, v) &= b_1 uv + b_2 u\bar{v} + b_3 \bar{u}v + b_4 \bar{u}\bar{v}. \end{aligned}\right\} \tag{2}$$

This transformation will also be called the transformation $\left.\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{matrix}\right\}$, and $a_1, \ldots a_4;\ b_1, \ldots b_4$ will be called the coefficients of the transformation.

In general, x and y cannot be considered as independent variables when they are submitted to this transformation. For, by eliminating u and v from (1) and (2), we deduce

$$p(a_1, x;\ b_1, y)\, p(a_2, x;\ b_2, y)\, p(a_3, x;\ b_3, y)\, p(a_4, x;\ b_4, y) = 0. \tag{3}$$

Thus x and y are limited to be pairs of roots of equation (3). Conversely, equation (3) is the condition of the possibility of equations (1) and (2). Thus, if (3) is satisfied, (1) and (2) are satisfied. Hence, (1) and (2) form the general solution of (3).

Thus the theory of this type of transformation is simply the theory of equations. A few theorems connected with it will be given incidentally in this part.

A distinct type of transformation exists when equation (3) is satisfied identically for all values of x and y. The condition for this is that each of the coefficients of xy, $x\bar{y}$, $\bar{x}y$, $\bar{x}\bar{y}$ in the development of the left-hand side of (3) should vanish. This condition can be exhibited in the single fundamental equation

$$\Pi(\bar{a}_r + \bar{b}_r) + \Pi(\bar{a}_r + b_r) + \Pi(a_r + b_r) + \Pi(a_r + b_r) = 0, \quad (r = 1, 2, 3, 4). \quad (4)$$

When this condition is satisfied, the transformation transforms the independent variables x and y into the independent variables u and v. Thus equations (1) and (2) can be solved for u and v in terms of x and y, and a reverse transformation is thus obtained of the form

$$u = F_1(x, y), \quad v = F_2(x, y), \quad (5)$$

where

$$\left.\begin{aligned} F_1(x, y) &= \alpha_1 xy + \alpha_2 x\bar{y} + \alpha_3 \bar{x}y + \alpha_4 \bar{x}\bar{y}, \\ F_2(x, y) &= \beta_1 xy + \beta_2 x\bar{y} + \beta_3 \bar{x}y + \beta_4 \bar{x}\bar{y}. \end{aligned}\right\} \quad (6)$$

Thus, this type of transformation is reversible. Let the term "substitution" be used exclusively for it.

In the case of substitutions, there is no advantage in changing the notations for the independent variables from x and y to u and v. Let a substitution T be defined to consist in the substitution of Tx for x and of Ty for y, where

$$\left.\begin{aligned} Tx &= a_1 xy + a_2 x\bar{y} + a_3 \bar{x}y + a_4 \bar{x}\bar{y}, \\ Ty &= b_1 xy + b_2 x\bar{y} + b_3 \bar{x}y + b_4 \bar{x}\bar{y}; \end{aligned}\right\} \quad (7)$$

and the coefficients of the substitution T satisfy equation (4). Also $T\phi(x, y)$ is defined to mean $\phi(Tx, Ty)$: for instance, $T\bar{x}$ means $\overline{}(Tx)$.

§2.—*Relations between the Coefficients of a Substitution.*

We have now to consider more particularly the implications of equation (4). Let us first eliminate b_1, b_2, b_3, b_4 from the equation; put $\lambda(b_1, b_2, b_3, b_4)$ for the left-hand side. We find that the only factors composing $\Pi\lambda\begin{pmatrix} i, & i, & i, & i \\ 0, & 0, & 0, & 0 \end{pmatrix}$ which do not equal i are the six of the type $\lambda(0, 0, i, i)$, $\lambda(i, 0, 0, i)$, etc. Also,

$$\lambda(0, 0, i, i) = a_1 a_2 + \bar{a}_1 \bar{a}_2 + a_3 a_4 + \bar{a}_3 \bar{a}_4,$$

with analogous values for the other factors of the same type. Hence,

$$\Pi\lambda\begin{pmatrix} i, & i, & i, & i \\ 0, & 0, & 0, & 0 \end{pmatrix} = \Sigma a_p\, a_q\, a_r + \Sigma \bar{a}_p\, \bar{a}_q\, \bar{a}_r, \quad (p,\, q,\, r = 1,\ 2,\ 3,\ 4),$$

where, as usual, p, q, r are unequal in the same product.

Hence, the resultant of (4), after eliminating b_1, b_2, b_3, b_4, is

$$\Sigma a_p\, a_q\, a_r + \Sigma \bar{a}_p\, \bar{a}_q\, \bar{a}_r = 0, \quad (p,\, q,\, r) = 1,\ 2,\ 3,\ 4). \tag{8}$$

A similar equation holds of b_1, b_2, b_3, b_4. Hence, the director functions, Tx and Ty, of any substitution are each of deficiency two and of supplemental deficiency two. Thus, functions of this type play a fundamental rôle in the theory of substitutions.

The conditions (8), which hold between the coefficients of a function of deficiency two and supplemental deficiency two, can be put otherwise thus: We have from (8),

$$a_1\, a_2\, (a_3 + a_4) = 0,$$

hence,
$$a_1\, a_2 \mathrel{\not\in} \bar{a}_3\, \bar{a}_4.$$

Also
$$\bar{a}_3\, \bar{a}_4\, (\bar{a}_1 + \bar{a}_2) = 0,$$

hence,
$$\bar{a}_3\, \bar{a}_4 \mathrel{\not\in} a_1\, a_2.$$

Thus
$$a_1\, a_2 = \bar{a}_3\, \bar{a}_4.$$

Hence, equation (8) can be replaced by the set of equations

$$a_p\, a_q = \bar{a}_r\, \bar{a}_s, \quad (p,\, q,\, r,\, s = 1,\ 2,\ 3,\ 4). \tag{9}$$

A similar set of equations holds for b_1, b_2, b_3, b_4.

Secondly, these relations (8) or (9), between the coefficients of each of the director functions separately, do not exhaust the implications of equation (4). For, from equations (9), we deduce the two sets,

$$\left.\begin{aligned} \Sigma a_p\, a_q\, \bar{b}_r\, \bar{b}_s &= \Sigma \bar{a}_p\, \bar{a}_q\, b_r\, b_s = \Sigma a_p\, a_q\, b_p\, b_q = \Sigma \bar{a}_p\, \bar{a}_q\, \bar{b}_p\, \bar{b}_q, \\ \Sigma a_p\, a_q\, b_r\, b_s &= \Sigma a_p\, a_q\, \bar{b}_p\, \bar{b}_q = \Sigma \bar{a}_p\, \bar{a}_q\, \bar{b}_r\, \bar{b}_s = \Sigma \bar{a}_p\, \bar{a}_q\, b_p\, b_q\,; \end{aligned}\right\} \tag{10}$$

where $(p,\, q,\, r,\, s = 1,\ 2,\ 3,\ 4)$.

Then, by the application to equation (4) of (8) and (10), we deduce

$$\Sigma a_p\, a_q\, b_p\, b_q + \Sigma \bar{a}_p\, \bar{a}_q\, b_p\, b_q = 0, \tag{11}$$

or equivalent forms deduced by the use of (10). A complete equivalent to equation (4) is given by

$$\Sigma\,(a_p a_q + \bar{a}_p \bar{a}_q)(b_p b_q + \bar{b}_p \bar{b}_q) = 0\,, \quad (p,\, q = 1,\, 2,\, 3,\, 4)\,; \tag{12}$$

that is,

$$\Sigma \bar{p}\,(a_p,\, a_q;\, b_p,\, b_q) = 0\,.$$

It is easy to see that this equation includes (11). It also includes (8), for we can deduce (8) from it by eliminating b_1, b_2, b_3, b_4.

Equation (8) may be solved for a_3 and a_4. The general solution is

$$\left.\begin{aligned} a_3 &= \bar{a}_1 \bar{a}_2 + p\,(\bar{a}_1 + \bar{a}_2)\,, \\ a_4 &= \bar{a}_1 \bar{a}_2 + \bar{p}\,(\bar{a}_1 + \bar{a}_2)\,. \\ \text{Similarly,} \qquad b_3 &= \bar{b}_1 \bar{b}_2 + q\,(\bar{b}_1 + \bar{b}_2)\,, \\ b_4 &= \bar{b}_1 \bar{b}_2 + \bar{q}\,(\bar{b}_1 + \bar{b}_2)\,. \end{aligned}\right\} \tag{13}$$

But a_1, a_2, b_1, b_2, and p and q, cannot be assumed arbitrarily consistently with the equation (4). For from (12) we find

$$(a_1 a_2 + \bar{a}_1 \bar{a}_2)(b_1 b_2 + \bar{b}_1 \bar{b}_2) = 0 \tag{14}$$

and

$$(a_1 \bar{a}_2 b_1 \bar{b}_2 + \bar{a}_1 a_2 \bar{b}_1 b_2)(pq + \bar{p}\bar{q}) + (a_1 \bar{a}_2 b_1 b_2 + \bar{a}_1 a_2 b_1 \bar{b}_2)(p\bar{q} + \bar{p}q) = 0. \tag{15}$$

Equation (15) does not require any relation to be fulfilled between a_1, a_2, b_1, b_2. Thus when a_1, a_2, a_3, a_4 have been chosen to satisfy (14), p and q can be chosen to satisfy (15), and then a_3, a_4, b_3, b_4 are given by equation (13). By this process the coefficients of any substitution can be found.

It is to be noticed that any three of a_1, a_2, b_1, b_2 can be chosen arbitrarily, and equation (14) can then be satisfied by a proper choice of the fourth. Also, either p or q can be assumed arbitrarily and equation (15) satisfied by a proper choice of the other.

Again, if a_1, a_2, a_3 have been chosen so as to satisfy

$$a_1 a_2 a_3 + \bar{a}_1 \bar{a}_2 \bar{a}_3 = 0, \tag{m}$$

then a_4 is definitely determined by the equation

$$a_4 = \bar{a}_2 \bar{a}_3 + \bar{a}_3 \bar{a}_1 + \bar{a}_1 \bar{a}_2. \tag{16}$$

For, solve the first of equations (13) to find the general value of p which is con-

sistent with the given values of a_1, a_2, a_3 and simplify by the use of (m). We find

$$p = a_1 a_3 + a_2 a_3 + u\,(a_3 + a_1 a_2).$$

Thence, from the second of equations (13), we find

$$a_4 = \bar{a}_1 \bar{a}_2 + \bar{a}_1 \bar{a}_3 + \bar{a}_2 \bar{a}_3 + u\,(\bar{a}_1 \bar{a}_3 + \bar{a}_2 \bar{a}_3 + \bar{a}_1 \bar{a}_2),$$

which reduces to equation (16). Hence, if three coefficients of a function $\phi\,(x,\,y)$ of deficiency two and of supplemental deficiency two are given, the function is completely determined. Thus, if the six coefficients, a_1, a_2, a_3, b_1, b_2, b_3 of a substitution are given, the substitution is completely determined.

§3.—*The Reverse Substitution.*

There is only one reverse substitution corresponding to a given substitution T; that is, there is only one substitution T' such that

$$T'Tx = x, \quad T'Ty = 0.$$

This proposition requires proof, for if we solve the equations

$$x = f_1\,(u,\,v), \quad y = f_2\,(u,\,v),$$

for u and v we find

$$u = F_1\,(x,\,y,\,q_1,\,q_2), \quad v = F_2\,(x,\,y,\,q_1,\,q_2),$$

where q_1 and q_2 are arbitraries which have been introduced in the solution, and each particular choice of q_1 and q_2 would seem to determine a different substitution which is reverse to T. We have to show that there is only one set of roots for u and v in terms of x and y.

The equations for u and v can be written

$$p\,(a_1,\,x)\,uv + p\,(a_2,\,x)\,u\bar{v} + p\,(a_3,\,x)\,\bar{u}v + p\,(a_4,\,x)\,\bar{u}\bar{v} = 0\,,$$
$$p\,(b_1,\,y)\,uv + p\,(b_2,\,y)\,u\bar{v} + p\,(b_3,\,y)\,\bar{u}v + p\,(b_4,\,y)\,\bar{u}\bar{v} = 0.$$

These two equations can be combined into

$$p\,(a_1,\,x\,;\;b_1,\,y)\,uv + p\,(a_2,\,x\,;\;b_2,\,y)\,u\bar{v} + p\,(a_3,\,x\,;\;b_3,\,y)\,\bar{u}v$$
$$+\,p\,(a_4,\,x\,;\;b_4,\,y)\,\bar{u}\bar{v} = 0\,.$$

The necessary and sufficient condition (cf. Part I, §3) that this equation for u and v should only have one set of roots is that the left-hand side should be a secondary linear prime. The condition for this is

$$\Sigma\bar{p}\,(a_q,\,x\,;\;b_q,\,y)\,\bar{p}\,(a_r,\,x\,;\;b_r,\,y) = 0, \quad (q,\,r = 1,\,2,\,3,\,4).$$

This equation can be written

$$\Sigma\,(a_q a_r x + \bar{a}_q \bar{a}_r \bar{x})(b_q b_r y + \bar{b}_q \bar{b}_r \bar{y}) = 0, \quad (q, r = 1, 2, 3, 4).$$

But from equation (12), this equation is satisfied identically for all values of x and y. Thus there is only one substitution reverse to T. Let it be written T^{-1}. It is evident that T is the reverse substitution to T^{-1}. Also, let the identical substitution be written T^0, so that

$$TT^{-1} = T^{-1}T = T^0.$$

Assume that

$$T^{-1}x = \alpha_1 xy + \alpha_2 x\bar{y} + \alpha_3 \bar{x}y + \alpha_4 \bar{x}\,\bar{y},$$
$$T^{-1}y = \beta_1 xy + \beta_2 x\bar{y} + \beta_3 \bar{x}y + \beta_4 \bar{x}\,\bar{y}.$$

Then, by hypothesis,

$$T^{-1}f_1(x, y) = x, \quad T^{-1}f_2(x, y) = y.$$

Hence, for all values of x and y,

$$x = f_1(\alpha_1, \beta_1)\,xy + f_1(\alpha_2, \beta_2)\,x\bar{y} + f_1(\alpha_3, \beta_3)\,\bar{x}y + f_1(\alpha_4, \beta_4)\,\bar{x}\,\bar{y},$$
$$y = f_2(\alpha_1, \beta_1)\,xy + f_2(\alpha_2, \beta_2)\,x\bar{y} + f_2(\alpha_3, \beta_3)\,\bar{x}y + f_2(\alpha_4, \beta_4)\,\bar{x}\,\bar{y}.$$

Hence,

$$\bar{f}_1(\alpha_1, \beta_1) = 0, \quad \bar{f}_1(\alpha_2, \beta_2) = 0, \quad f_1(\alpha_3, \beta_3) = 0, \quad f_1(\alpha_4, \beta_4) = 0,$$
$$\bar{f}_2(\alpha_1, \beta_1) = 0, \quad f_2(\alpha_2, \beta_2) = 0, \quad \bar{f}_2(\alpha_3, \beta_3) = 0, \quad f_2(\alpha_4, \beta_4) = 0.$$

Hence, α_1 and β_1 satisfy the equation

$$\bar{f}_1(u, v) + \bar{f}_2(u, v) = 0;$$

and α_2 and β_2 satisfy the equation

$$\bar{f}_1(u, v) + f_2(u, v) = 0;$$

and α_3 and β_2 satisfy the equation

$$f_1(u, v) + \bar{f}_2(u, v) = 0;$$

and α_4 and β_4 satisfy the equation

$$f_1(u, v) + f_2(u, v) = 0.$$

By the use of equations (8) and (10) and (11), it is easy to verify that the left-hand side of each of these equations is a secondary linear prime (cf. Part I, equation (4)) and, therefore, each equation has only one set of roots. Thus we obtain another proof of the previous proposition. Again, if $\phi(x, y)$ is a secondary linear prime, the solution of $\phi(x, y) = 0$ is

$$x = CD, \quad y = BD.$$

Hence, the solution for the coefficients of the reverse substitution are

$$\left.\begin{aligned} \alpha_1 &= (\bar{a}_3 + \bar{b}_3)(\bar{a}_4 + \bar{b}_4), & \beta_1 &= (\bar{a}_2 + \bar{b}_2)(\bar{a}_4 + \bar{b}_4), \\ \alpha_2 &= (\bar{a}_3 + b_3)(\bar{a}_4 + b_4), & \beta_2 &= (\bar{a}_2 + b_2)(\bar{a}_4 + b_4), \\ \alpha_3 &= (a_3 + \bar{b}_3)(a_4 + \bar{b}_4), & \beta_3 &= (a_2 + \bar{b}_2)(a_4 + \bar{b}_4), \\ \alpha_4 &= (a_3 + b_3)(a_4 + b_4), & \beta_4 &= (a_2 + b_2)(a_4 + b_4) \end{aligned}\right\} \tag{17}$$

An interesting example of a substitution is

$$\left.\begin{aligned} Tx &= p\,(a, x), \\ Ty &= p\,(b, y). \end{aligned}\right\} \tag{18}$$

It is easy to verify that in this case

$$T = T^{-1}, \text{ that is, } T^2 = T^0.$$

§4.—*The Group of Substitutions.*

If T_1 and T_2 are any two substitutions, then $T_1\,T_2$ and $T_2\,T_1$ are substitutions (as distinct from non-reversible transformations), also we have proved that one, and only one, reverse substitution corresponds to any given substitution. Hence, substitutions form a group.

The group of substitutions is not continuous since the concepts of quantity and of infinitesimal variations of quantities have no place among the concepts of this algebra. It is of finite order, if the number of distinct terms in the algebra representing given fundamental constants from which all reasoning starts, is conceived as finite. It is of indefinite order in so far as we may always suppose new constants to be produced without violating any of the laws of the algebra.

If T be any substitution the subgroup T, T^2, T^3, ... is necessarily of finite order, and is therefore, cyclical. For only a finite number of terms can be generated by algebraic combinations of the coefficients of T, and the coefficients of T^2 T^3, ... must be selections from these coefficients. The order can be reduced by the assumption of additional relations among the coefficients of T. Thus, in the substitution of equation (18), the order is two.

§5.—*Substitutions Satisfying Special Conditions.*

To find the condition which the functions $\psi_1(x, y)$, $\psi_2(x, y)$, $\psi_3(x, y)$, $\psi_4(x, y)$, must satisfy, if the coefficients of some substitution T are such that, with the

usual notation, a_1 and b_1 are a pair of roots of $\psi_1(x, y)$, a_2 and b_2 of $\psi_2(x, y)$, a_3 and b_3 of $\psi_3(x, y)$, a_4 and b_4 of $\psi_4(x, y)$. These equations can be expressed in the typical form

$$\psi_r(a_r, b_r) = 0, \quad (r = 1, 2, 3, 4) \ldots. \tag{19}$$

Let
$$\psi_r(x, y) = F_r xy + G_r x\bar{y} + H_r \bar{x}y + K_r \bar{x}\bar{y}.$$

The complete condition satisfied by a_r, b_r, $(r = 1, 2, 3, 4)$ is found by combining equations (4) and (19). It becomes.

$$\Pi(\bar{a}_r + \bar{b}_r) + \Pi(\bar{a}_r + b_r) + \Pi(a_r + \bar{b}_r) + \Pi(a_r + b_r) + \Sigma\psi_r(a_r, b_r) = 0, \tag{20}$$
$$(r = 1, 2, 3, 4).$$

We have to eliminate a_r, b_r $(r = 1, 2, 3, 4)$ from this equation. Put $\lambda(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ for its left-hand side. Now,

$$\begin{array}{llllll}
\Pi(\bar{a}_r + \bar{b}_r) = i, & \text{unless at least one pair,} & a_r = i, & b_r = i, \\
\Pi(\bar{a}_r + b_r) = i, & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} & a_r = i, & b_r = 0, \\
\Pi(a_r + \bar{b}_r) = i, & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} & a_r = 0, & b_r = i, \\
\Pi(a_r + b_r) = i, & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} & a_r = 0, & b_r = 0,
\end{array}$$

Hence, each of the factors composing $\Pi\lambda\begin{pmatrix} i, & i, & i, & i, & i, & i, & i, & i \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}$ is i, except those in which simultaneously $a_p = i$, $b_p = i$; $a_q = i$, $b_q = 0$; $a_r = 0$, $b_r = i$, $a_s = 0$, $b_s = 0$; where $(p, q, r, s = 1, 2, 3, 4)$.

Hence the resultant of equation (20) is

$$\Pi(F_p + G_q + H_r + K_s) = 0, \quad (p, q, r, s = 1, 2, 3, 4). \tag{21}$$

This is the required condition which the coefficients of equations (19) must satisfy.

We can deduce as special cases of the above, the following theorems:

The condition which the coefficients of the first three of equations (19) must satisfy in order that (a_1, b_1) may be chosen to satisfy the first, (a_2, b_2) the second, and (a_3, b_3) the third, is deduced from equation (21) by putting zero for F_4, G_4, H_4, K_4.

The condition which the coefficients of the first two of equation (19) must satisfy in order that (a_1, b_1) may be chosen to satisfy the first and (a_2, b_2) the second, is found from equation (21) by putting zero for F_3, G_3, H_3, K_3 and also

for F_4, G_4, H_4, K_4. It can be written out in full in the form

$$F_1 G_1 H_1 K_1 + F_2 G_2 H_2 K_2 + F_1 G_1 H_1 . F_2 G_2 H_2 + F_1 G_1 K_1 . F_2 G_2 K_2 \\ + F_1 H_1 K_1 . F_2 H_2 K_2 + G_1 H_1 K_1 . G_2 H_2 K_2 = 0. \quad (22)$$

This condition in addition to securing that the two equations

$$\psi_1 (x, y) = 0, \ \psi_2 (x y) = 0,$$

are possible, also secures that the function $\psi_1 (x, y) \psi_2 (x, y)$ is of supplemental deficiency two. This latter part of the condition is satisfied if either of the functions is of supplemental deficiency two. For instance both (a_1, b_1) and (a_2, b_2) can be chosen to be pairs of roots of the same equation

$$\psi_1 (x, y) = 0,$$

if $\psi_1 (x, y)$ is of supplemental deficiency two. Similarly from the previous theorem it follows that (a_1, b_1), (a_2, b_2), (a_3, b_3) can be chosen to be pairs of roots of the same equation, $\psi_1 (x, y) = 0$, if $\psi_1 (x, y)$ is of supplemental deficiency three. But it follows from (21) that it is impossible that (a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4) should be pairs of roots of the same equation.

This fact, that there are sets of four pairs of terms which cannot be severally pairs of roots of the same equation of two variables, is one of the fundamental facts of the algebra. The analogue for equations of one variable is that the two terms a and $\bar{a}$ cannot be both roots of the same equation of one variable. This fundamental fact is the correlative of the fact that there are functions which are not evanescible.

Again, if we assume that a_2 and b_2 are to satisfy the second of equations (19) and a_3 and b_3 the third, and a_4 and b_4 the fourth of equations (19), let us investigate the conditions which are thereby imposed on a_1 and b_1. We have to eliminate a_2, b_2, a_3, b_3, a_4, b_4, from these three equations and from equation (4); and the resulting equation for a_1, b_1 will be the required condition. It will be more convenient to substitute the equivalent equation (12) for equation (4). Thus the complete equation from which the elimination is to be performed can be written

$$\Sigma (a_1 a_p + \bar{a}_1 \bar{a}_p) (b_1 b_p + \bar{b}_1 \bar{b}_p) + \Sigma (a_p a_q + \bar{a}_p \bar{a}_q) (b_p b_q + \bar{b}_p \bar{b}_q) \\ + \Sigma \psi_p (a_p, b_p) = 0, \quad (p, q = 2, 3, 4).$$

Now put $\lambda (a_2, a_3, a_4, b_2, b_3, b_4)$ for the left-hand side of this equation. Then any of the factors composing $\Pi \lambda \begin{pmatrix} i, & i, & i, & i, & i, & i \\ 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}$, for which either $a_2 = a_3$, and $b_2 = b_3$

simultaneously, or $a_2 = a_4$ and $b_2 = b_4$ simultaneously, or $a_3 = a_4$ and $b_3 = b_4$ simultaneously, it has the value i. Accordingly considering the factors which involve none of these three possibilities, we find that the required condition is

$$\Pi\,(G_p + H_q + K_r)\,a_1\,b_1 + \Pi\,(F_p + H_q + K_r)\,a_1\,\bar{b}_1 + \Pi\,(F_p + G_q + K_r)\,\bar{a}_1\,b_1 \\ + \Pi\,(F_p + G_q + H_r)\,\bar{a}_1\,\bar{b}_1 = 0, \; (p, q, r = 2, 3, 4). \quad (23)$$

Thus if $a_1\,b_1$ is also to satisfy the first of equations (19), the complete condition which it satisfies is

$$\{F_1 + \Pi\,(G_p + H_q + K_r)\}\,a_1 b_1 + \{G_1 + \Pi\,(F_p + H_q + K_r)\}\,a_1\bar{b}_1 \\ + \{H_1 + \Pi\,(F_p + G_q + K_r)\}\,\bar{a}_1 b_1 + \{K_1 + \Pi\,(F_p + G_q + H_r)\}\,\bar{a}_1\bar{b}_1 = 0. \quad (24)$$

The condition for the possibility of this equation is simply equation (21), as should evidently be the case.

§6.—*Congruence of Functions.*

Two functions $\phi\,(x, y)$ and $\Phi\,(x, y)$ are said to be congruent, if one or more substitutions T exist, such that

$$T\phi\,(x, y) = \Phi\,(x, y).$$

The relation of congruence will be expressed by the symbolism

$$\phi\,(x, y) \longleftrightarrow \Phi\,(x, y). \quad (25)$$

It is evident that if

$$\phi\,(x, y) \longleftrightarrow \Phi\,(x, y),$$

and

$$\Phi\,(x, y) \longleftrightarrow \psi\,(x, y),$$

then

$$\phi\,(x, y) \longleftrightarrow \psi\,(x, y).$$

Thus the relation is transitive.

A set of functions such that any two are congruent, and which comprises all the functions congruent to members of the set, is called a congruent family.

We will now prove the fundamental theorem that a congruent family is composed of all the functions with a given set of invariants.

For let

$$\phi\,(x, y) = Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\,\bar{y},$$
$$\Phi\,(x, y) = Fxy + Gx\bar{y} + H\bar{x}y + K\bar{x}\,\bar{y}.$$

We require the necessary and sufficient condition that

$$\phi\,(x, y) \longleftrightarrow \Phi\,(x, y).$$

Now with the usual notation for the coefficients of the substitution T, we have

$$T\phi(x, y) = \phi(a_1, b_1)\,xy + \phi(a_2, b_2)\,x\bar{y} + \phi(a_3, b_3)\,\bar{x}y + \phi(a_4, b_4)\,\bar{x}\,\bar{y}.$$

Hence $\phi(a_1, b_1) = F,\ \phi(a_2, b_2) = G,\ \phi(a_3, b_3) = H,\ \phi(a_4, b_4) = K.$

These equations can be written

$$\left.\begin{aligned}
p(A, F)\,a_1b_1 + p(B, F)\,a_1\bar{b}_1 + p(C, F)\,\bar{a}_1b_1 + p(D, F)\,\bar{a}_1\bar{b}_1 &= 0,\\
p(A, G)\,a_2b_2 + p(B, G)\,a_2\bar{b}_2 + p(C, G)\,\bar{a}_2b_2 + p(D, G)\,\bar{a}_2\bar{b}_2 &= 0,\\
p(A, H)\,a_3b_3 + p(B, H)\,a_3\bar{b}_3 + p(C, H)\,\bar{a}_3b_3 + p(D, H)\,\bar{a}_3\bar{b}_3 &= 0,\\
p(A, K)\,a_4b_4 + p(B, K)\,a_4\bar{b}_4 + p(C, K)\,\bar{a}_4b_4 + p(D, K)\,\bar{a}_4\bar{b}_4 &= 0.
\end{aligned}\right\} \qquad (26)$$

Now by comparison with equations (19) and (21), we see that the requisite condition is

$$\Pi p(A, F;\ B, G;\ C, H;\ D, K) = 0, \qquad (27)$$

where, in order to obtain the various factors, F, G, H, K are kept in the same positions, and A, B, C, D are permuted in every possible way, so that each appears in each factor.

In order to effect the solution of this equation, it will be convenient to alter the notation. Consider

$$\Pi p(x_1, c_p;\ x_2, c_q;\ x_3, c_r;\ x_4, c_s) = 0, \quad (p, q, r, s = 1, 2, 3, 4). \qquad (28)$$

Let C_1, C_2, C_3, C_4 be the symmmetric functions of c_1, c_2, c_3, c_4; and let X_1, X_2, X_3, X_4 be the symmetric functions of x_1, x_2, x_3, x_4.

Now, consider any one factor of the left-hand side of (28); for example, $p(x_1, c_p;\ x_2, c_q;\ x_3, c_r;\ x_4, c_s)$.

Then

$$\begin{aligned}
p(x_1, c_p;\ x_2, c_q;\ x_3, c_r;\ x_4, c_s) = {}& \bar{C}_4x_1x_2x_3x_4 + \Sigma\,[(\bar{c}_p + \bar{c}_q + c_r + c_s)\,x_1x_2x_3\bar{x}_4]\\
&+ \Sigma\,[(\bar{c}_p + \bar{c}_q + c_r + c_s)\,x_1x_2\bar{x}_3\bar{x}_4] + \Sigma\,[(\bar{c}_p + c_q + c_r + c_s)\,x_1\bar{x}_2\bar{x}_3\bar{x}_4] + C_1\bar{x}_1\bar{x}_2\bar{x}_9\bar{x}_4.
\end{aligned}$$

Hence, equation (28) becomes

$$\bar{C}_4X_4 + {}^{-}(C_3\bar{C}_4)\,X_3\bar{X}_4 + {}^{-}(C_2\bar{C}_3)\,X_2\bar{X}_3 + {}^{-}(C_1\bar{C}_2)\,X_1\bar{X}_2 + C_1\bar{X}_1 = 0.$$

But, from the symmetry of (28), it is obvious that we may equally well write

$$\bar{X}_4C_4 + {}^{-}(X_3\bar{X}_4)\,C_3\bar{C}_4 + {}^{-}(X_2\bar{X}_3)\,C_2\bar{C}_3 + {}^{-}(X_1\bar{X}_2)\,C_1\bar{C}_2 + X_1\bar{C}_1 = 0.$$

By adding these two equations, and remembering that

$$P\bar{Q} + \bar{P}Q = 0,$$

implies $P = Q$, we find

$$X_4 = C_4,\ X_3\overline{X_4} = C_3\overline{C_4},\ X_2\overline{X_3} = C_2\overline{C_3},\ X_1\overline{X_2} = C_1\overline{C_2},\ X_1 = C_1. \qquad \text{(m)}$$

The fourth and fifth of equations (m) give

$$C_1\overline{X_2} = C_1\overline{C_2}.$$

Hence, $$X_2 = C_2 + q\overline{C_1}.$$

But $$X_2\overline{C_1} = X_2\overline{X_1} = 0,$$

and $$C_2\overline{C_1} = 0.$$

Hence, $$q\overline{C_1} = 0.$$

Thus $X_2 = C_2$. An exactly similar proof applied to the third of equations (m) will now prove that $X_3 = C_3$.

Hence, equation (27) is equivalent to the fact that the corresponding invariants of $\phi(x, y)$ and $\Phi(x, y)$ are equal.

As special examples of this theorem, we note that (1) secondary linear primes form a congruent family, (2) secondary separable primes form a congruent family, (3) functions both of deficiency two and of supplemental deficiency two form a congruent family. This last family includes as members all primary primes whether functions of x only or of y only.

A congruent family is entirely defined by its invariants. Accordingly, we shall name a family by its invariants, and speak of the family (S_1, S_2, S_3, S_4).

Any family (S_1, S_2, S_3, S_4) includes as a member the function

$$S_1xy + S_2\bar{x}y + S_3x\bar{y} + S_4\bar{x}\bar{y},$$

which can also be written

$$S_1xy + S_2x + S_3y + S_4;$$

and also the family includes the twenty-three other functions of the same type found by permuting the invariants in their use as coefficients.

Call such functions the "canonical functions" of the family.

There is no fundamental distinction in property between the various canonical functions of a family. Accordingly, we shall habitually use the one mentioned above, and will call it "the canonical function." Thus the canonical function of the family $(i, i, i, 0)$, which is the family of secondary linear primes, is $x + y$. The canonical function of the family $(i, 0, 0, 0)$ which is the family of secondary separable primes, is xy. The canonical function of the family

$(i, i, 0, 0)$ is x. The only families which contain some linear functions as some members, have been proved (cf. Part I, §9) to be those for which $S_1 = S_2$. Thus the canonical function for such a family is $S_1 x + S_3 y + S_4$. Similarly, the only families which contain some separable functions as some members are those for for which $S_3 = S_4$. Thus the canonical function for such a family is

$$S_1 xy + S_2 x + S_4, \text{ that is, } (x + S_4)(S_1 y + S_2).$$

Also, by reference to equations (23) and (24), we can find from equations (26) the complete conditions satisfied by a_1 and b_1 after the other coefficients have been eliminated. For let S_{11}, S_{12}, S_{13} be the symmetric functions of B, C, D; and let S_{21}, S_{22}, S_{23} be the symmetric functions of A, C, D, and S_{31}, S_{32}, S_{33} of A, B, D and S_{41}, S_{42}, S_{43} of A, B, C; and let R_{11}, R_{12}, R_{13} be the symmetric functions of G, H, K, and so on. Now, consider $\Pi p(B, G; C, H; D, K)$, where the various factors are formed by keeping G, H, K in their places and permuting B, C, D. By comparison with the evaluation of the left-hand side of (28), we see that

$$\Pi p(B, G; C, H; D, K)$$
$$= p(S_{13}, R_{13}; S_{12}\overline{S}_{13}, R_{12}\overline{R}_{13}; S_{11}\overline{S}_{12}, R_{11}\overline{R}_{12}; S_{11}, R_{11}),$$

with similar expressions for other similar products. Thus we find the required equation to be

$$\begin{aligned} &p(A, F; S_{13}, R_{13}; S_{12}\overline{S_{13}}, R_{12}\overline{R}_{13}; S_{11}\overline{S}_{12}, R_{11}\overline{R}_{12}; S_{11}, R_{11})\, a_1 b_1 \\ &\quad + p(B, F; S_{23}, R_{13}; S_{22}\overline{S}_{23}, R_{12}\overline{R}_{13}; S_{21}\overline{S}_{22}, R_{11}\overline{R}_{12}; S_{21}, R_{11})\, a_1 \overline{b}_1 \\ &\quad + p(C, F; S_{33}, R_{13}; S_{32}\overline{S}_{33}, R_{12}\overline{R}_{13}; S_{31}\overline{S}_{32}, R_{11}\overline{R}_{12}; S_{31}, R_{11})\, \overline{a}_1 b_1 \\ &\quad + p(D, F; S_{43}, R_{13}; S_{42}\overline{S}_{43}, R_{12}\overline{R}_{13}; S_{41}\overline{S}_{42}, R_{11}\overline{R}_{12}; S_{41}, R_{11})\, \overline{a}_1 \overline{b}_1 = 0. \quad (29) \end{aligned}$$

Also a similar equation can be found for a_2, b_2 by putting G for F and R_{21}, R_{22}, R_{23} for R_{11}, R_{12}, R_{13} respectively in equation (29). Also, similarly for a_3, b_3 and for a_4, b_4 by similar interchanges.

Thus a_1, b_1 can be chosen to be any pair of roots of equation (29), but when a_1, b_1 is once chosen, a_2, b_2 and a_3, b_3 and a_4, b_4 must be suitable pairs of roots of their corresponding equations.

It is easily proved by considering the invariants, that if

$$\phi_1(x, y) \longleftrightarrow \Phi_1(x, y)$$

and

$$\phi_2(x, y) \longleftrightarrow \Phi_2(x, y),$$

then, for all values of λ,

$$\lambda\phi_1(x, y) + \bar{\lambda}\phi_2(x, y) \leftrightarrow \lambda\Phi_1(x, y) + \bar{\lambda}\Phi_2(x, y).$$

The conditions that $\phi(x, y)$ can be transformed into $\Phi(x, y)$ by some transformation which is not necessarily a substitution, are given by the resultants of equations (26) without the use of equation (4). These conditions are, if S_1, S_2, S_3, S_4 are the invariants of $\phi(x, y)$ and R_1, R_2, R_3, R_4 of $\Phi(x, y)$,

$$\bar{S}_1 F + S_4\bar{F} = 0, \quad \bar{S}_1 G + S_4\bar{G} = 0, \quad \bar{S}_k H + S_4\bar{H} = 0,$$
$$\bar{S}_1 K + S_4\bar{K} = 0.$$

Hence, by addition,

$$\bar{S}_1 R_1 + S_4\bar{R}_4 = 0.$$

Thus the required conditions can be written

$$R_1 \mathrel{\not\!\!\leftarrow\!\!=} S_1, \quad S_4 \mathrel{\not\!\!\leftarrow\!\!=} R_4. \tag{30}$$

In other words, the field of $\phi(x, y)$ must contain the field* of $\Phi(x, y)$. Thus the conditions that $\phi(x, y)$ can be transformed into $\Phi(x, y)$, and that $\Phi(x, y)$ can be transformed into $\phi(x, y)$, are

$$R_1 = S_1, \quad R_4 = S_4. \tag{31}$$

Accordingly, the additional conditions required in order that these transformations may be substitutions, are

$$R_2 = S_2, \quad R_3 = S_2.$$

§7.—*The Identical Group of a Function.*

Any function $\phi(x, y)$ can be conceived as congruent to itself. Also the substitutions such that

$$T\phi(x, y) = \phi(x, y)$$

evidently form a group. For, if T is such a substitution, T^{-1} also has the same property, and if T_1 is another such substitution, then $T_1 T$ has also the same property. Let this group be called the "identical group" of the function $\phi(x, y)$.

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* Cf. "Universal Algebra," §33.

By substituting A, B, C, D for F, G, H, K in equations (26), we see that the coefficients of a substitution of the identical group must satisfy the equations

$$\left.\begin{aligned} * + p(B, A)\, a_1 \bar{b}_1 + p(C, A)\, \bar{a}_1 b_1 + p(D, A)\, \bar{a}_1 \bar{b}_1 &= 0, \\ p(A, B)\, a_2 b_2 + * + p(C, B)\, \bar{a}_2 b_2 + p(D, B)\, \bar{a}_2 \bar{b}_2 &= 0, \\ p(A, C)\, a_3 b_3 + p(B, C)\, a_3 \bar{b}_3 + * + p(D, C)\, \bar{a}_3 \bar{b}_3 &= 0, \\ p(A. D)\, a_4 b_4) p(B, D)\, a_4 \bar{b}_4 + p(C, D)\, \bar{a}_4 b_4 + * &= 0, \end{aligned}\right\} \quad (32)$$

The coefficients must also, of course, satisfy equation (4).

Then equation (29), in the reduced form required for substitutions of the identical group of $\phi(x, y)$, becomes

$$\begin{aligned} &p(B, A;\ S_{23}, S_{13};\ S_{22} \bar{S}_{23}, S_{12} \bar{S}_{13};\ S_{21} \bar{S}_{22}, S_{11} \bar{S}_{12};\ S_{21}, S_{11})\, \bar{a}_1 b_1 \\ &\quad + p(C, A;\ S_{33}, S_{13};\ S_{32} \bar{S}_{33}, S_{12} \bar{S}_{13};\ S_{31} \bar{S}_{32}, S_{11} \bar{S}_{12};\ S_{31}, S_{11})\, a_1 \bar{b}_1 \\ &\quad + p(D, A;\ S_{43}, S_{13};\ S_{42} \bar{S}_{43}, S_{12} \bar{S}_{13};\ S_{41} \bar{S}_{42}, S_{11} \bar{S}_{12};\ S_{41}, S_{11})\, \bar{a}_1 \bar{b}_1 = 0; \quad \text{(m)} \end{aligned}$$

with similar equations for a_2, b_2 and for a_3, b_3, and for a_4, b_4.

Now $\qquad S_{11} = B + C + D,\ S_{12} = BC + BD + CD,\ S_{13} = BCD,$

with similar meanings for analogous terms. Hence it is easily seen that

$$\begin{aligned} &p(S_{23}, S_{13}) = CD\, p(A, B),\ p(S_{22} \bar{S}_{23}, S_{12} \bar{S}_{13}) = p(A, B), \\ &p(S_{21} \bar{S}_{22}, S_{11} \bar{S}_{12}) = p(A, B),\ p(S_{21}, S_{11}) = \bar{C} \bar{D}\, p(A, B). \end{aligned}$$

Thus

$$p(B, A;\ S_{23}, S_{13};\ S_{22} \bar{S}_{23}, S_{12} \bar{S}_{13};\ S_{21} \bar{S}_{22}, S_{11} \bar{S}_{12};\ S_{21}, S_{11}) = p(A, B);$$

with similar simplifications for the coefficients of $a_1 \bar{b}_1$ and $\bar{a}_1 \bar{b}_1$. Thus equation (m) reduces to the first of equation (32). Similarly for the equations for a_2, b_2 and for a_3, b_3, and for a_4, b_4. Thus in order to find substitutions of the identical group of $\phi(x, y)$, a_1 and b_1 can be chosen to be any pair of roots of the first of equations (32), and then a_2, b_3 and a_3, b_3, and a_4, b_4 must be chosen to be suitable roots of their corresponding equations in (32). Or we may start from a_2, b_2 or from a_3, b_3, or from a_4, b_4.

The identical group of every function is of order greater than one, that is contains substitutions other than the identical substitution.

For if the identical group of $\phi(x, y)$ is of order one the left-hand side of every equation of the set (32) must be a secondary linear prime, so as to give only one pair of roots for a_1, b_1, and one for a_2, b_2, and so on. Now, each of the left-hand sides of equation (32) lacks one term, thus the first of these equations lacks

the term $a_1 b_1$, and so on. Hence [cf. Part I, §3], the left-hand sides can only be secondary linear primes if each remaining coefficient is i.

Thus every equation of the type

$$p(A, B) = i,$$

must hold. But this gives

$$p(\overline{A}, B) = 0, \text{ that is, } A = \overline{B}.$$

Similarly $A = \overline{C}$, $B = \overline{C}$. But these equations are inconsistent. Hence the identical group of $\phi(x, y)$ cannot be of order one.

The identical groups of all members of a congruent family are simply isomorphic.

For, let $\phi(x, y)$ and $\Phi(x, y)$ be two congruent functions; and let the members of the identical group of $\phi(x, y)$ be written T_ϕ, T'_ϕ, etc., and those of the identical group of $\Phi(x, y)$ be written T_Φ, T'_Φ, etc. Also let T be any substitution such that

$$T\phi(x, y) = \Phi(x, y).$$

Then TT_ϕ, TT'_ϕ, ... are evidently such substitutions.

Similarly $T_\Phi T$, $T'_\Phi T$, ... are such substitutions. Again let T_1 be another substitution such that

$$T_1 \phi(x, y) = \Phi(x, y),$$

Then $$T^{-1} T_1 \phi(x, y) = T^{-1}\Phi(x, y) = \phi(x, y).$$

Hence $T^{-1} T_1$ is a member of the identical group of $\phi(x, y)$. Accordingly we may write

$$T^{-1} T_1 = T_\phi, \text{ that is, } T_1 = T T_\phi.$$

Similarly $$T_1 = T_\Phi T.$$

Thus each of the sets $T T_\phi$, $T T'_\phi$, ..., and $T_\Phi T$, $T'_\Phi T$, ... includes every substitution which turns $\phi(x, y)$ into $\Phi(x, y)$. Accordingly, by a proper choice of T_ϕ, or of T_Φ, we can always write

$$T T_\phi = T_\Phi T.$$

Hence $$T_\phi = T^{-1} T_\Phi T.$$

Hence the identical groups of $\phi(x, y)$ and of $\Phi(x, y)$ are simply isomorphic. Accordingly it is only necessary to study the structure of the identical group of one member of a congruence family; for instance, that of the canonical form.

For instance, let us investigate the general expression for a substitution of the identical group of the canonical function of the family $(i, i, 0, 0)$. This canonical function is x. Thus if T is the required substitution, we have

$$Tx = x,\ Ty = b_1 xy + b_2 \bar{x}y + \bar{b}_3 xy + b_4 \bar{x}\bar{y}.$$

Hence from equation (12),

$$b_1 b_2 + \bar{b}_1 \bar{b}_2 = 0,\ b_3 b_4 + \bar{b}_3 \bar{b}_4 = 0.$$

Thus $$Ty = xp(\bar{b}_1, y) + \bar{x}p(\bar{b}_3, y); \tag{33}$$

where b_1 and b_3 can be assumed arbitrarily. Thus the general form for a substitution of the group is found.

Now let T and T' be two substitutions of this group, so that

$$Tx = x,\ Ty = b_1 xy + \bar{b}_1 x\bar{y} + b_3 \bar{x}y + \bar{b}_3 \bar{x}\bar{y},$$
$$T'x = x,\ T'y = b'_1 xy + \bar{b}'_1 x\bar{y} + b'_3 \bar{x}y + \bar{b}'_3 \bar{x}\bar{y}.$$

Then

$$T'Ty = \bar{p}(b_1, b'_1)\, xy + p(b_1, b'_1)\, x\bar{y} + \bar{p}(b_3, b'_3)\bar{x}y + p(b_3, b'_3)\bar{x}\bar{y} = TT'y. \tag{34}$$

Hence the substitutions TT' and $T'T$ are the same. Thus this identical group is Abelian.

Also in equation (34), put $b'_1 = b_1$, and $b'_3 = b_3$; we find

$$T^2 y = y.$$

Thus $T^2 = T^0$. Hence every substitution of the group is of order two.

The equations satisfied by the coefficients of any substitution of the identical group of the canonical function of the family (S_1, S_2, S_3, S_4), are found from equations (32) to be

$$\left.\begin{aligned} * + S_1\bar{S}_2 a_1 \bar{b}_1 + S_1\bar{S}_3 \bar{a}_1 b_1 + S_1\bar{S}_4 \bar{a}_1 \bar{b}_1 \qquad &= 0,\\ S_1\bar{S}_2 a_2 b_2 + * + S_2\bar{S}_3 \bar{a}_2 b_2 + S_2\bar{S}_4 \bar{a}_2 \bar{b}_2 &= 0,\\ S_1\bar{S}_3 a_3 b_3 + S_2\bar{S}_3 a_3 \bar{b}_3 + * + S_3\bar{S}_4 \bar{a}_3 \bar{b}_3 &= 0,\\ S_1\bar{S}_4 a_4 b_4 + S_2\bar{S}_4 a_4 b_4 + S_3\bar{S}_4 \bar{a}_4 b_4 + * &= 0, \end{aligned}\right\} \tag{35}$$

together with equations (4); and it has been proved that any one pair, such as a_1 and b_1, can be assumed to be any pair of roots of their corresponding equation.

§8.—*Common Subgroups of Identical Groups.*

The identical groups of any two functions have a common subgroup which always includes other substitutions in addition to the identical substitution,

except in the case when the two functions are the two director functions of a substitution.

For let $\phi(x, y)$ and $\Phi(x, y)$ be the two functions, where A, B, C, D are the coefficients of $\phi(x, y)$, and F, G, H, K are the coefficients of $\Phi(x, y)$. Then the coefficients of any substitution common to the two identical groups must satisfy two sets of equations of the type of (32). These two sets can be combined into the single set

$$\left.\begin{array}{l} * + p(B, A;\ G, F)\, a_1 \bar{b}_1 + p(C, A;\ H, F)\, \bar{a}_1 b_1 + p(D, A;\ K, F)\, \bar{a}_1 \bar{b}_1 = 0, \\ p(A, B;\ F, G)\, a_2 b_2 + * + p(C, B;\ H, G)\, \bar{a}_2 b_2 + p(D, B;\ K, G)\, \bar{a}_2 \bar{b}_2 = 0, \\ p(A, C;\ F, H)\, a_3 b_3 + p(B, C;\ G, H)\, \bar{a}_3 b_3 + * + p(D, C;\ K, H)\, \bar{a}_3 \bar{b}_3 = 0, \\ p(A, D;\ F, K)\, a_4 b_4 + p(B, C;\ G, K)\, \bar{a}_4 b_4 + p(C, D;\ H, K)\, a_4 \bar{b}_4 + * = 0, \end{array}\right\} \quad (36)$$

and, in addition, equation (4) must be satisfied.

Now, by reasoning in all respects the same as that in the previous article, if there is only one substitution satisfying these conditions, the left-hand side of each of equation (36) is a secondary linear prime. Hence, every equation of the type

$$\bar{p}(B, A;\ G, F) = 0,$$

must hold. But this typical equation is

$$(AB + \bar{A}\bar{B})(F\bar{G} + \bar{F}G) = 0.$$

Hence, by comparison with equation (12), we see that this condition requires that $\phi(x, y)$ and $\Phi(x, y)$ should be a pair of director functions of a substitution. The above proposition can be stated thus: The Identical Group of any function $\phi(x, y)$ which does not belong to the family $(i, i, 0, 0)$ has a subgroup containing more than the one member T^0 in common with the identical group of any other function whatever. The same proposition is true of any function $\phi(x, y)$ which does belong to the family $(i, i, 0, 0)$, except in those cases when the second function also belongs to the same family $(i, i, 0, 0)$, and, in addition, is so related to $\phi(x, y)$ that the two functions are the director functions of a substitution.

It has been proved [cf. Part I, §8, equation* (24)] that we can always write

$$\phi(x, y) = S_4 + S_1(\bar{S}_4 + U_1)\,\phi'(x, y),$$

* A misprint in equation (24), Part I, is here corrected.

where the coefficients of $\phi'(x, y)$ are given by equations (25) of Part I, and the invariants by equations (26) of Part I.

Now $\qquad T\phi(x, y) = S_4 + S_1(\overline{S}_4 + U_1)\, T\phi'(x, y).$

Hence, if $\qquad T\phi'(x, y) = \phi'(x, y),$

it follows that $\qquad T\phi(x, y) = \phi(x, y).$

Hence, the identical group of $\phi'(x, y)$ is a subgroup of the identical group of $\phi(x, y)$. Now, the invariants of $\phi'(x, y)$ are given [cf. equation (26) of §8 Part I] by

$$\left.\begin{aligned} S'_1 &= S_1\overline{S}_4 + S_1U_1 + V_1, \\ S'_2 &= S_2\overline{S}_4 + S_2U_2 + \overline{S}_1V_2 + \overline{S}_4U_1V_2, \\ S'_3 &= S_3\overline{S}_4 + S_3U_3 + \overline{S}_1V_3 + S_4\overline{U}_1V_3, \\ S'_4 &= S_4U_4 + \overline{S}_1V_4 + S_4\overline{U}_1V_4, \end{aligned}\right\} \tag{37}$$

and U_1, U_2, U_3, U_4 are the symmetric functions of u_1, u_2, u_3, u_4 and V_1, V_2, V_3, V_4 of v_1, v_2, v_3, v_4.

Also, if $\phi(x, y)$ is the canonical function of the family (S_1, S_2, S_3, S_4), then equations (25) of §8, Part I, become

$$\begin{aligned} A' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_1 + (\overline{S}_4 + u_1)\, S_1, \\ B' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_2 + (\overline{S}_4 + u_2)\, S_2, \\ C' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_3 + (\overline{S}_4 + u_3)\, S_3, \\ D' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_4 + (\overline{S}_4 + u_4)\, S_3. \end{aligned}$$

Hence, in general, $\phi'(x, y)$ does not become the canonical function of (S'_1, S'_2, S'_3, S'_4). But if we make

$$\begin{aligned} &u_1 = U_1, \quad u_2 = U_2, \quad u_3 = U_3, \quad u_4 = U_4, \\ &v_1 = V_1, \quad v_2 = V_2, \quad v_3 = V_3, \quad v_4 = V_4. \end{aligned}$$

which can be done without altering U_1, U_2, U_3, U_4 or V_1, V_2, V_3, V_4, then $\phi'(x, y)$ becomes the canonical function of the family (S'_1, S'_2, S'_3, S'_4). Hence, the identical group of the canonical function of the family (S'_1, S'_2, S'_3, S'_4) is a subgroup of the identical group of the canonical function of the family (S_1, S_2, S_3, S_4).

For instance, put

$$\begin{aligned} &U_1 = U_2 = i, \quad U_3 = U_4 = 0, \\ &V_1 = V_2 = i, \quad V_3 = V_4 = 0. \end{aligned}$$

We deduce that the identical group of the canonical function of the family $(i, S_2 + \bar{S}_1, S_3 \bar{S}_4, 0)$ is a subgroup of the identical group of the canonical function of the family $S_1, S_2, S_3, S_4)$.

Now, it is proved in §9 of Part I, that if $S_1 = S_2$ and $S_3 = S_4$, the family contains both linear members and separable members. Also, in this case,

$$S_2 + \bar{S}_1 = i, \quad S_3 \bar{S}_4 = 0,$$

and the canonical function of the family $(i, i, 0, 0)$ is x. Hence, the identical group of x is a subgroup of the identical groups of the canonical functions of all families which contain both linear and separable members. Thus, from the discussion at the end of the previous paragraph, all these identical groups have a common Abelian subgroup.

Secular Perturbations of the Planets.

By G. W. Hill.

Gauss first clearly indicated the rôle elliptic functions play in this subject.* Halphen has since presented the investigation in a very elegant manner.† The modifications made by the latter in the procedure of Gauss are chiefly the transference of the origin of rectangular coordinates to the attracted planet, and, instead of the differential of the eccentric anomaly, the adoption of the element of area described by the radius of the disturbing planet expressed in terms of the differentials of the rectangular coordinates. He also appeals to the qualities of the cone formed by the orbit of the attracting planet as contour of base and the position of the attracted planet as vertex; this improvement, however, had been previously indicated by Bour.‡ A remarkable degree of elegance is attained by these changes; but it seems to me that additional statements are needed to show the connection with the astronomical problem which originally suggested the investigation; for Halphen, like Gauss, treats only the attraction of a certain form of ring. This, of course, is to ignore the second integration which the problem demands. Perhaps, therefore, I shall be pardoned if I here attempt to supply the mentioned lack.

In the fashion of Halphen we take the attracted planet as the origin of rectangular coordinates, but the orientation of the axes is, for the present, left indeterminate. The coordinates of the attracting planet we denote by x, y, z; and the coordinates of the Sun, which are the negatives of those of the attracted planet referred to the Sun, will be x_0, y_0, z_0. Let ρ be the distance of the attracting planet from the origin, so that $\rho^2 = x^2 + y^2 + z^2$; and let g denote the planet's

* Gauss, Werke, vol. III, pp. 331–355.

† G. H. Halphen, Traité des Fonctions Elliptiques et de leurs Applications, Tom. II, pp. 310–328.

‡ Journal de l'École Polytechnique, Cahier XXXVI, pp. 59-84.

mean anomaly. Then the secular perturbations of the attracted planet depend on the three definite integrals

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{x}{\rho^3}\,dg\,,\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{y}{\rho^3}\,dg\,,\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{z}{\rho^3}\,dg\,.$$

But while g is passing from 0 to 2π, the area described by the radius of the planet augments from 0 to πab, if a and b are severally the major and minor semi-axes. Thus, if σ denote this varying area, the preceding integrals may be written

$$\frac{1}{\pi ab}\int_0^{\pi ab}\frac{x}{\rho^3}\,d\sigma\,,\quad \frac{1}{\pi ab}\int_0^{\pi ab}\frac{y}{\rho^3}\,d\sigma\,,\quad \frac{1}{\pi ab}\int_0^{\pi ab}\frac{z}{\rho^3}\,d\sigma\,.$$

The tetrahedron, with $d\sigma$ as base and the origin as vertex, has, for volume one sixth of the following expression:

$$6\,V = x_0\,(ydz - zdy) + y_0\,(zdx - xdz) + z_0\,(xdy - ydx).$$

But, if h denote the perpendicular from the origin on the plane of the orbit of the attracting planet, we also have $3\,V = hd\sigma$. Hence

$$d\sigma = \frac{x_0}{2h}\,(ydz - zdy) + \frac{y_0}{2h}\,(zdx - xdz) + \frac{z_0}{2h}\,(xdy - ydx)\,.$$

Then, if we derive the quantities P_x, P_y, P_z, etc., from integrating the expressions

$$dP_x = \tfrac{1}{2}\,\frac{x\,(ydz - zdy)}{\rho^3}\,,\qquad dP_y = \tfrac{1}{2}\,\frac{y\,(ydz - zdy)}{\rho^3}\,,\qquad dP_z = \tfrac{1}{2}\,\frac{z\,(ydz - zdy)}{\rho^3}\,,$$

$$dQ_x = \tfrac{1}{2}\,\frac{x\,(zdx - xdz)}{\rho^3}\,,\qquad dQ_y = \tfrac{1}{2}\,\frac{y\,(zdx - xdz)}{\rho^3}\,,\qquad dQ_z = \tfrac{1}{2}\,\frac{z\,(zdx - xdz)}{\rho^3}\,,$$

$$dR_x = \tfrac{1}{2}\,\frac{x\,(xdy - ydx)}{\rho^3}\,,\qquad dR_y = \tfrac{1}{2}\,\frac{y\,(xdy - ydx)}{\rho^3}\,,\qquad dR_z = \tfrac{1}{2}\,\frac{z\,(xdy - ydx)}{\rho^3}\,,$$

around the whole orbit, the integrals above, which we will denote by X, Y, Z will be given by the following expressions:

$$X = \frac{1}{\pi abh}\,(x_0P_x + y_0Q_x + z_0R_x)\,,$$

$$Y = \frac{1}{\pi abh}\,(x_0P_y + y_0Q_y + z_0R_y)\,,$$

$$Z = \frac{1}{\pi abh}\,(x_0P_z + y_0Q_z + z_0R_z)\,.$$

The nine quantities involved in these expressions and obtained through integration are homogeneous and of the dimension zero with respect to the linear unit.

Moreover, if, in the cone formed by the orbit of the attracting planet as directrix and the origin as vertex, the plane of the base is shifted in any manner whatever, these nine quantities remain unchanged. For example, consider the expressions

$$\frac{x\,(ydz - zdy)}{\rho^3}, \quad \frac{y\,(ydz - zdy)}{\rho^3}, \quad \frac{z\,(ydz - zdy)}{\rho^3}.$$

If we put

$$x = \rho \sin\theta, \quad y = \rho\cos\theta\cos\lambda, \quad z = \rho\cos\theta\sin\lambda,$$

they are transformed into

$$\sin\theta\cos^2\theta\, d\lambda, \quad \cos^3\theta\cos\lambda\, d\lambda, \quad \cos^3\theta\sin\lambda\, d\lambda$$

Let the equation of the cone be

$$Ax^2 + By^2 + Cz^2 + Dyz + Fzx + Fxy = 0,$$

or

$$A\tan^2\theta + B\cos^2\lambda + C\sin^2\lambda + D\sin\lambda\cos\lambda + E\tan\theta\sin\lambda + F\tan\theta\cos\lambda = 0.$$

Since, from this equation, θ is obtainable as a function of λ, it is evident that the integrals of the preceding differential expressions, extended to the whole course of variation of λ, depend solely on the elements of the cone and are altogether independent of the plane section called the base.

A simple addition of the differentials shows that

$$P_x + Q_y + R_z = 0.$$

Also we have

$$d(Q_z - R_y) = \tfrac{1}{2}\, d\left(\frac{x}{\rho}\right), \quad d(R_x - P_z) = \tfrac{1}{2}\, d\left(\frac{y}{\rho}\right), \quad d(P_y - Q_x) = \tfrac{1}{2}\, d\left(\frac{z}{\rho}\right).$$

But the second members of these equations, integrated along the orbit to the point of beginning, give zero as the result: hence

$$Q_z = R_y, \quad R_x = P_z, \; P_y = Q_x.$$

Thus

$$X = \frac{1}{\pi abh}\,(x_0 P_x + y_0 Q_x + z_0 P_z),$$

$$Y = \frac{1}{\pi abh}\,(x_0 Q_x + y_0 Q_y + z_0 R_y),$$

$$Z = \frac{1}{\pi abh}\,(x_0 P_z + y_0 R_y + z_0 R_z).$$

And, if we put

$$\Phi = \frac{1}{2\pi abh}\left[x_0^2 P_x + y_0^2 Q_y + z_0^2 R_z + 2x_0 y_0 Q_x + 2y_0 z_0 R_y + 2z_0 x_0 P_z\right],$$

we shall have

$$X = \frac{\partial\Phi}{\partial x_0}, \quad Y = \frac{\partial\Phi}{\partial y_0}, \quad Z = \frac{\partial\Phi}{\partial z_0}.$$

The orientation of the axes of coordinates which serve to define the variables x, y, z, x_0, y_0, z_0 has been left undetermined; but now suppose that the axes of symmetry of the cone are employed for this purpose. Then the equation of the cone takes the form

$$\frac{x^2}{G_x} + \frac{y^2}{G_y} + \frac{z^2}{G_z} = 0,$$

G_x, G_y, G_z being constants of which two are of one sign and the other of the opposite sign. Plainly, if this equation is satisfied by the set of values x, y, z, it is satisfied by any of the eight sets $\pm x, \pm y, \pm z$. Consequently, each positive element of the six quantities $Q_z, R_y, R_x, P_z, P_y, Q_x$ is accompanied by a corresponding negative element. Thus, in this case, these quantities vanish. With this selection of axes we, therefore, have

$$X = \frac{x_0}{\pi abh} P_x, \quad Y = \frac{y_0}{\pi abh} Q_y, \quad Z = \frac{z_0}{\pi abh} R_z.$$

The naming of the coordinates is, of course, arbitrary, but, to settle the choice, we suppose that G_x, G_y, G_z are in the order of algebraic magnitude, the first and second being negative, while the last is positive. The equation of the cone appears to involve three variables, but, as we may divide the left member by the square of any one of them, it is, in reality, a relation between two variables; thus, but one variable can be regarded as independent. The equation is then satisfied if we make

$$x = \varepsilon\sqrt{-G_x}\cos T, \quad y = \varepsilon\sqrt{-G_y}\sin T, \quad z = \varepsilon\sqrt{G_z},$$

where ε is an indeterminate which disappears when the substitution is made in dP_x, dQ_y, dR_z, and T is the new variable introduced by Gauss and may be regarded as indicating the position of the planet in its orbit; its function, in this respect, being precisely similar to those fulfilled by the mean, eccentric and true anomalies, and it may thus, with propriety, be designated as a *perspective anomaly.* When T goes from 0 to 2π, the planet makes a complete circuit of its orbit.

The substitution made, we have

$$\begin{aligned} ydz - zdy &= -\,\varepsilon^2\sqrt{-\,G_y G_z}\cos T dT, \\ zdx - xdz &= -\,\varepsilon^2\sqrt{-\,G_x G_z}\sin T dT, \\ xdy - ydx &= \varepsilon^2\sqrt{G_x G_y}\,dT, \\ \rho^2 &= \varepsilon^2\left[G_z - G_y\sin^2 T - G_x\cos^2 T\right]. \end{aligned}$$

The quantities G_x, G_y, G_z are usually determined in such a way that $G_x G_y G_z = a^2 b^2 h^2$; also we may introduce k the modulus of the elliptic integrals involved and m such that

$$k^2 = \frac{G_y - G_x}{G_z - G_x}, \quad m = \sqrt{G_z - G_x}.$$

Then our integrals take the forms

$$X = -\,\frac{x_0}{m^3}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^2 T dT}{(1 - k^2\sin^2 T)^{\frac{3}{2}}}, \quad Y = -\,\frac{y_0}{m^3}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^2 T dT}{(1 - k^2\sin^2 T)^{\frac{3}{2}}},$$

$$Z = \frac{z_0}{m^3}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{dT}{(1 - k^2\sin^2 T)^{\frac{3}{2}}}.$$

The methods of evaluating these definite integrals severally proposed by Legendre, Gauss and Jacobi have all about the same degree of rapid convergence but that of the last is to be preferred because it expresses the values explicitly in terms of a parameter q called the *nome*. Putting $k = \sin\theta$, q can be derived from the equation

$$\frac{q + q^9 + q^{25} + \ldots}{1 + 2(q^4 + q^{16} + q^{36} + \ldots)} = \left[\frac{\sin\frac{\theta}{2}}{1 + \sqrt{\cos\theta}}\right]^2.$$

The solution is most readily accomplished by the method of tentation. Then, if we adopt two functions of q, K and L such that

$$K = \frac{4}{\cos^2\theta\,(1 + \sqrt{\cos\theta})^2}\left[1 + 2(q^4 + q^{16} + \ldots)\right]^2,$$

$$L = \frac{(1 + \sqrt{\cos\theta})^3}{\sin^2\theta\cos^{\frac{5}{2}}\theta}\,\frac{q - 4q^4 + 9q^9 - 16q^{16} + \ldots}{\left[1 + 2(q^4 + q^{16} + \ldots)\right]^3},$$

we have

$$X = -\,\frac{x_0}{m^3}\,L\cos^2\theta, \quad Y = \frac{y_0}{m^3}(L - K), \quad Z = \frac{z_0}{m^3}(K - L\sin^2\theta).*$$

It will be seen from these expressions that, if X, Y, Z are regarded as the

* For the proof of these formulas, reference may be made to Bertrand, Calcul Intégral, Liv. III., Chap. VII.

components along the axes of coordinates of a force acting on the attracted planet, one elliptic integral suffices for determining the orientation of the resultant, but that an additional one is required if the magnitude of the latter is to be found. It will be an advantage, therefore, if instead of tabulating K and L as functions of k, q or θ, we take two other quantities M and $\varkappa$, such that

$$M = K - L\sin^2\theta, \quad \sin^2\varkappa = \frac{L\cos^2\theta}{K - L\sin^2\theta}, \quad \cos^2\varkappa = \frac{K - L}{K - L\sin^2\theta}.$$

Then we shall have

$$X = -\frac{M}{m^3}\sin^2\varkappa . x_0, \quad Y = -\frac{M}{m^3}\cos^2\varkappa . y_0, \quad Z = \frac{M}{m^3} z_0 .$$

Let R denote the magnitude of the resultant, H and Λ severally the latitude and longitude of the point in the heavens towards which it is directed; the circles of reference being the principal axes of the sphero-conic traced in the heavens by the frequently mentioned cone. Also let r_0 denote the distance of the Sun from the attracted planet and η_0, λ_0 severally its latitude and longitude referred to the same circles. Then our equations will stand

$$R\cos H\sin\Lambda = -\frac{M}{m^3}\sin^2\varkappa . r_0\cos\eta_0\sin\lambda_0,$$

$$R\sin H = -\frac{M}{m^3}\cos^2\varkappa . r_0\sin\eta_0,$$

$$R\cos H\cos\Lambda = \frac{M}{m^3} . r_0\cos\eta_0\cos\lambda_0 .$$

If we put

$$N = \frac{M}{m^3} r_0\cos\eta_0,$$

we shall have

$$\tan\Lambda = -\sin^2\varkappa\tan\lambda_0, \quad \tan H = -\frac{\cos^2\varkappa\tan\eta_0\cos\Lambda}{\cos\lambda_0}, \quad R = \frac{N\cos\lambda_0}{\cos H\cos\Lambda}.$$

As, except in very particular cases, it is not easy to select at the outset the axes of symmetry of the cone for the adopted axes of coordinates, we must find the position of these axes in reference to another system which is known. The equation of the cone having a very complicated expression when the axes of coordinates are quite general, we select a particular system such that the treatment may be as easy as possible. Let the axis of x have the orientation of the line going from the centre of the ellipse described by the attracting planet to its

perihelion, that of y the orientation of the line going from the centre to the point where the eccentric anomaly is 90° and that of z the orientation of the line going from the centre to the north pole of the plane of the orbit. Let the coordinates of the centre of the ellipse be, in their order, A, B, C; we prefer these designations although, with e as the eccentricity and the previous notation, they have the equivalents

$$A = x_0 - ae, \quad B = y_0, \quad C = z_0.$$

Then the equations of the orbit of the attracting planet are

$$\left(\frac{x-A}{a}\right)^2 + \left(\frac{y-B}{b}\right)^2 = 1. \quad z = C.$$

In order to have the equation of the cone so frequently mentioned it is only necessary to multiply the several terms of the first equation by factors selected from the equivalent quantities $1 = \frac{z}{C} = \frac{z^2}{C^2}$, in such a way that they may all become homogeneous and of two dimensions in x, y, z. Thus the equation of the cone may be written in the shape

$$\frac{z^2}{C^2} - \left[\frac{x - \frac{A}{C}z}{a}\right]^2 - \left[\frac{y - \frac{B}{C}z}{b}\right]^2 = 0.$$

This equation is not, in general, referred to the axes of symmetry, hence we proceed to make the linear transformation of variables which will bring this about. Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ denote the rectangular coordinates referred to the symmetric axes of the cone, and write the formulas of transformation thus:

$$\begin{aligned}
\mathrm{X} &= \alpha x + \beta y + \gamma z, & \alpha^2 + \beta^2 + \gamma^2 &= 1, & \alpha\alpha' + \beta\beta' + \gamma\gamma' &= 0, \\
\mathrm{Y} &= \alpha' x + \beta' y + \gamma' z, & \alpha'^2 + \beta'^2 + \gamma'^2 &= 1, & \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' &= 0, \\
\mathrm{Z} &= \alpha'' x + \beta'' y + \gamma'' z, & \alpha''^2 + \beta''^2 + \gamma''^2 &= 1, & \alpha''\alpha + \beta''\beta + \gamma''\gamma &= 0,
\end{aligned}$$

the inverse of which are

$$\begin{aligned}
x &= \alpha\mathrm{X} + \alpha'\mathrm{Y} + \alpha''\mathrm{Z}, & \alpha^2 + \alpha'^2 + \alpha''^2 &= 1, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' &= 0, \\
y &= \beta\mathrm{X} + \beta'\mathrm{Y} + \beta''\mathrm{Z}, & \beta^2 + \beta'^2 + \beta''^2 &= 1, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' &= 0, \\
z &= \gamma\mathrm{X} + \gamma'\mathrm{Y} + \gamma''\mathrm{Z}, & \gamma^2 + \gamma'^2 + \gamma''^2 &= 1, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' &= 0.
\end{aligned}$$

The equation of the cone, after substitution of the new variables, should take the form

$$\frac{\mathrm{X}^2}{G_x} + \frac{\mathrm{Y}^2}{G_y} + \frac{\mathrm{Z}^2}{G_z} = 0,$$

where G_x, G_y, G_z are constants and functions of the quantities A, B, C, a, b. They are the roots of a certain cubic which may be obtained in the following way: Let $V = 0$ denote the first form of the equation of the cone, and set

$$\frac{x}{G} = \frac{1}{2}\frac{\partial V}{\partial x}, \quad \frac{y}{G} = \frac{1}{2}\frac{\partial V}{\partial y}, \quad \frac{z}{G} = \frac{1}{2}\frac{\partial V}{\partial z}.$$

From these equations, which are linear in x, y, z, eliminate these variables; the result is a cubic in G whose roots are the values of G_x, G_y, G_z. In the special form of V with which we have to deal, these equations, after a slight modification, can be thus stated:

$$\frac{x}{G} = \frac{A}{C}\frac{z}{G+a^2}, \quad \frac{y}{G} = \frac{B}{C}\frac{z}{G+b^2}, \quad \frac{z}{G} = \frac{z}{C^2} - \frac{A}{C}\frac{x}{G} - \frac{B}{C}\frac{y}{G}.$$

The elimination of x, y, z from these equations gives

$$\frac{A^2}{G+a^2} + \frac{B^2}{G+b^2} + \frac{C^2}{G} = 1.$$

It is known that the roots of this equation in G are all real, two of them being negative and one positive. The mode of assignment of these roots as values of G_x, G_y, G_z has already been described. Thus we are enabled to discover the values of the latter without reference to the nine coefficients α, β, γ, etc. But in our further progress we shall need to know the latter. They are readily found from the first and second equations of the penultimate group combined with the conditions $\alpha^2 + \beta^2 + \gamma^2 = 1$, etc. In the first we make $x = \alpha$, $y = \beta$, $z = \gamma$ and set G_x for G; again make $x = \alpha'$, $y = \beta'$, $z = \gamma'$ and set G_y for G; lastly make $x = \alpha''$, $y = \beta''$, $z = \gamma''$ and set G_z for G. Thus we have the equations

$$\alpha = \frac{A}{C}\frac{G_x}{G_x+a^2}\gamma, \quad \beta = \frac{B}{C}\frac{G_x}{G_x+b^2}\gamma,$$

$$\gamma^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_x}{G_x+a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_x}{G_x+b^2}\right)^2},$$

$$\alpha' = \frac{A}{C}\frac{G_y}{G_y+a^2}\gamma', \quad \beta' = \frac{B}{C}\frac{G_y}{G_y+b^2}\gamma',$$

$$\gamma'^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_y}{G_y+a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_y}{G_y+b^2}\right)^2},$$

$$\alpha'' = \frac{A}{C}\frac{G_z}{G_z+a^2}\gamma'', \quad \beta'' = \frac{B}{C}\frac{G_z}{G_z+b^2}\gamma'',$$

$$\gamma''^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_z}{G_z+a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_z}{G_z+b^2}\right)^2}.$$

These equations furnish the values of the nine coefficients of the transformation. The signs of the three γ's are indeterminate: one may take them as positive.

The cubic in G furnishes us with the relations

$$\frac{A^2}{G_x + a^2} + \frac{B^2}{G_x + b^2} + \frac{C^2}{G_x} = 1,$$
$$\frac{A^2}{G_y + a^2} + \frac{B^2}{G_y + b^2} + \frac{C^2}{G_y} = 1,$$
$$\frac{A^2}{G_z + a^2} + \frac{B^2}{G_z + b^2} + \frac{C^2}{G_z} = 1.$$

Regarding A^2, B^2, C^2 as the unknowns in these equations, their solution gives

$$A^2 = \frac{(G_x + a^2)(G_y + a^2)(G_z + a^2)}{a^2(a^2 - b^2)},$$
$$B^2 = \frac{(G_x + b^2)(G_y + b^2)(G_z + b^2)}{b^2(b^2 - a^2)},$$
$$C^2 = \frac{G_x G_y G_z}{a^2 b^2}.$$

By means of these values we can eliminate A, B, C from the foregoing expressions for the coefficients of transformation, and thus obtain values depending only on the five quantities G_x, G_y, G_z, a, b. We have thus

$$\alpha^2 = \frac{1}{a^2(a^2 - b^2)} \frac{G_x(G_x + b^2)(G_y + a^2)(G_z + a^2)}{(G_x - G_y)(G_x - G_z)},$$
$$\beta^2 = \frac{1}{b^2(b^2 - a^2)} \frac{G_x(G_x + a^2)(G_y + b^2)(G_z + b^2)}{(G_x - G_y)(G_x - G_z)},$$
$$\gamma^2 = \frac{1}{a^2 b^2} \frac{G_y G_z (G_x + a^2)(G_x + b^2)}{(G_x - G_y)(G_x - G_z)}.$$

The values of α'^2, β'^2, γ'^2 and again of α''^2, β''^2, γ''^2, are obtained from the preceding by simply making a cyclical permutation of the subscripts attached to the G; firstly from x, y, z into y, z, x, secondly from x, y, z into z, x, y. On account of the divisor $a^2 - b^2$, small when the eccentricity of the planet's orbit is small, some of these formulas labor under a disadvantage.

In computing the values of the definite integrals

$$\frac{1}{\pi ab}\int \frac{x}{\rho^3}\,d\sigma, \quad \frac{1}{\pi ab}\int \frac{y}{\rho^3}\,d\sigma, \quad \frac{1}{\pi ab}\int \frac{z}{\rho^3}\,d\sigma,$$

where x, y, z are referred to the system of axes determined by those of the ellipse described by the attracting planet, we may suppose that the four quantities x_0, y_0, z_0, h are simply constants and unaffected by the transformation we have just made to pass from the system of coordinates x, y, z to that of X, Y, Z. If, for the sake of discrimination, we designate the components along the mentioned axes as X', Y', Z', reserving X, Y, Z for the components along the symmetric axes, we evidently have

$$X' = \alpha X + \alpha' Y + \alpha'' Z, \quad Y' = \beta X + \beta' Y + \beta'' Z, \quad Z' = \gamma X + \gamma' Y + \gamma'' Z.$$

The components of the right members are determined the moment we have solved the cubic in G, for which the coordinates A, B, C furnish the requisite data. We have only to make

$$x_0 = A + ae, \quad y_0 = B, \quad z_0 = C, \quad h = \sqrt{C^2},$$

where h is always to be taken positively. Thus

$$X' = -\alpha\,(A + ae)\,\frac{M}{m^3}\sin^2 x - \alpha' B\,\frac{M}{m^3}\cos^2 x + \alpha'' C\,\frac{M}{m^3},$$
$$Y' = -\beta\,(A + ae)\,\frac{M}{m^3}\sin^2 x - \beta' B\,\frac{M}{m^3}\cos^2 x + \beta'' C\,\frac{M}{m^3},$$
$$Z' = -\gamma\,(A + ae)\,\frac{M}{m^3}\sin^2 x - \gamma' B\,\frac{M}{m^3}\cos^2 x + \gamma'' C\,\frac{M}{m^3}.$$

Substituting for α, α', α'', β, β', β'' their values in terms of γ, γ', γ'', if we put

$$U_x = -\frac{A + ae}{m^3}\,\frac{G_x \gamma}{C}\,M \sin^2 x,$$
$$U_y = -\frac{B}{m^3}\,\frac{G_y \gamma'}{C}\,M \cos^2 x,$$
$$U_z = \frac{C}{m^3}\,\frac{G_z \gamma''}{C}\,M,$$

we shall have

$$X' = A\left[\frac{U_x}{G_x + a^2} + \frac{U_y}{G_y + a^2} + \frac{U_z}{G_z + a^2}\right],$$
$$Y' = B\left[\frac{U_x}{G_x + b^2} + \frac{U_y}{G_y + b^2} + \frac{U_z}{G_z + b^2}\right],$$
$$Z' = C\left[\frac{U_x}{G_x} + \frac{U_y}{G_y} + \frac{U_z}{G_z}\right].$$

By substituting in U_x, U_y, U_z, the last values we have given for γ, γ', γ'', and bearing in mind that

$$G_z - G_x = m^2, \quad G_y - G_x = m^2 \sin^2\theta, \quad G_z - G_y = m^2\cos^2\theta,$$

the expressions for the first mentioned quantities become

$$U_x = \frac{A+ae}{m^5}\sqrt{G_x(G_x+a^2)(G_x+b^2)}\,\frac{M\sin^2 x}{\sin\theta},$$

$$U_y = \frac{B}{m^5}\sqrt{-G_y(G_y+a^2)(G_y+b^2)}\,\frac{M\cos^2 x}{\sin\theta\cos\theta},$$

$$U_z = \frac{C}{m^5}\sqrt{G_z(G_z+a^2)(G_z+b^2)}\,\frac{M}{\cos\theta},$$

where, if γ, γ', γ'' have been taken positively, the three radicals must receive the sign of C.

Let it be required to find the component of this attracting force directed towards the centre of the ellipse described by the attracting planet. Putting $\mathrm{r}^2 = A^2 + B^2 + C^2$, we ought to multiply the components given above severally by the factors $\frac{A}{\mathrm{r}}$, $\frac{B}{\mathrm{r}}$, $\frac{C}{\mathrm{r}}$, and take the sum. Which, if we do, and have regard to relations, G_x, G_y, G_z satisfy as being the roots of the cubic in G; calling this component Δ, we have

$$\mathrm{r}\,\Delta = U_x + U_y + U_z.$$

But the component R_0 directed towards the Sun will be more important. Putting r_0 for the radius vector of the attracted planet, we have

$$r_0^2 = (A+ae)^2 + B^2 + C^2.$$

Then the components X, Y, Z ought to be multiplied severally by $\frac{A+ae}{r_0}$, $\frac{B}{r_0}$, $\frac{C}{r_0}$, and thus is obtained

$$r_0 R_0 = U_x + U_y + U_z + aeX'.$$

In fact, by multiplying X', Y', Z' severally by the three systems of three multipliers

$$\begin{array}{ccc}
\dfrac{A+ae}{r_0}, & \dfrac{B}{r_0}, & \dfrac{C}{r_0}, \\[2ex]
-\dfrac{B}{\sqrt{r_0^2-C^2}}, & \dfrac{A+ae}{\sqrt{r_0^2-C^2}}, & 0, \\[2ex]
-\dfrac{(A+ae)\,C}{r_0\sqrt{r_0^2-C^2}}, & -\dfrac{BC}{r_0\sqrt{r_0^2-C^2}}, & \dfrac{\sqrt{r_0^2-C^2}}{r_0},
\end{array}$$

we shall arrive at a set of components, of which the first, already given, is directed toward the Sun, and the second is perpendicular thereto and lying in the plane of the attracting planet's orbit, while the third is perpendicular to both the preceding. If we call these new components X'', Y'', Z'', and if the angle between the planes of the orbits of the two planets be denoted by I, we shall have

$$R_0 = X'', \quad S_0 = Y'' \cos I - Z'' \sin I, \quad W_0 = Y'' \sin I + Z'' \cos I,$$

where R_0 is the component towards the Sun, S_0 the component perpendicular thereto and lying in the plane of the attracted planet's orbit, and W_0 the component perpendicular to that plane. But, if we make use of a rectangular set of elements instead of a polar, it is more commodious to refer the components to the plane and line of nodes of the attracted planet on the attracting planet's orbit. Let this ascending node be distant an arc $= \omega$ from the perihelion of the latter. Then the desired components X''', Y''', Z''' will be

$$\begin{aligned} X''' &= X' \cos \omega - Y' \sin \omega, \\ Y''' &= X' \cos I \sin \omega + Y' \cos I \cos \omega - Z' \sin I, \\ Z''' &= X' \sin I \sin \omega + Y' \sin I \cos \omega + Z' \cos I. \end{aligned}$$

It is often interesting to know the position of the great circles forming the principal axes of the sphero-conic in reference to the great circle marked out in the heavens by the plane of the attracting planet's orbit. If we call ☊ the longitude of the ascending node of one of the planes of symmetry of the cone on the plane xy, and i the inclination (always between 0° and 180°), and τ the angular distance of the centre of the sphero-conic from the node measured in the direction of increasing longitudes, the four following equations can be used for the determination of these quantities:

$$\tan i \sin \Omega = -\frac{A}{C}\,\frac{G_z}{G_z + a^2}, \quad \tan i \cos \Omega = \frac{B}{C}\,\frac{G_z}{G_z + b^2},$$

$$\sin i \sin \tau = -\left[1 + \frac{A^2}{C^2}\left(\frac{G_x}{G_x + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_x}{G_x + b^2}\right)^2\right]^{-\frac{1}{2}},$$

$$\sin i \cos \tau = -\left[1 + \frac{A^2}{C^2}\left(\frac{G_y}{G_y + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_y}{G_y + b^2}\right)^2\right]^{-\frac{1}{2}},$$

where the signs of the two radicals may be taken positive or negative, thus corresponding to the four intersections of the two great circles with the one great circle.

To find the two semi-axes of the sphero-conic as measured by the arcs they subtend in the heavens, take the equation of the cone

$$\frac{X^2}{G_x} + \frac{Y^2}{G_y} + \frac{Z^2}{G_z} = 0\,;$$

the section of the cone by the plane $Z = \sqrt{G_z}$ gives the ellipse whose equation is

$$-\frac{X^2}{G_x} - \frac{Y^2}{G_y} = 1\,.$$

The semi-axis of this in the direction of the axis of X is $\sqrt{-G_x}$ and in the direction of the axis of Y, $\sqrt{-G_y}$. The greatest latitude η_0 of the planet moving on the sphero-conic and the greatest longitude λ_0 will be given by the equations

$$\tan\eta_0 = \sqrt{-\frac{G_y}{G_z}}, \qquad \tan\lambda_0 = \sqrt{-\frac{G_x}{G_z}}\,.$$

The general equation connecting the variables η and λ will be obtained if, in the equation of the cone, we make

$$X = \rho\cos\eta\sin\lambda, \quad Y = \rho\sin\eta, \quad Z = \rho\cos\eta\cos\lambda,$$

and thus is

$$\frac{\sin^2\lambda}{G_x} + \frac{\tan^2\eta}{G_y} + \frac{\cos^2\lambda}{G_z} = 0\,.$$

These variables η and λ are expressed in terms of the perspective anomaly T as follows:

$$\cos\eta\cos\lambda = \varepsilon\sqrt{G_z}, \quad \cos\theta\sin\lambda = \varepsilon\sqrt{-G_x}\cos T, \quad \sin\eta = \varepsilon\sqrt{-G_y}\sin T.$$

We see that when $T = 0$, also $\eta = 0$ and $\tan\lambda = \sqrt{-\frac{G_x}{G_z}}$; and when $T = \frac{\pi}{2}$, $\tan\eta = \sqrt{-\frac{G_y}{G_z}}$, and $\lambda = 0$; when $T = \pi$, $\eta = 0$, $\tan\lambda = -\sqrt{\frac{G_x}{G_z}}$.

On the Solution of the Cubic in G and the Argument to be Employed in Tabulating the Elliptic Integrals.

A few words may be added in reference to the solution of the cubic in G,

$$\frac{A^2}{G + a^2} + \frac{B^2}{G + b^2} + \frac{C^2}{G} = 1\,.$$

Using r^2 for $A^2 + B^2 + C^2$, this equation expanded is

$$G^3 - [\mathrm{r}^2 - (a^2 + b^2)]\, G^2 + [a^2A^2 + b^2B^2 - (a^2 + b^2)\,\mathrm{r}^2 + a^2b^2]\, G - a^2b^2C^2 = 0\,.$$

It will be noticed that in obtaining the values of k, the modulus of the elliptic integrals involved, and of m, we do not need all the roots of this equation, but only their differences. Hence, it will be advantageous to put

$$G = J + \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)]\,, \quad \mathrm{a} = a^2 + \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)]\,,$$
$$\mathrm{b} = b^2 + \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)]\,, \quad \mathrm{c} = \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)]$$

The cubic will then take the form

$$\frac{A^2}{J + \mathrm{a}} + \frac{B^2}{J + \mathrm{b}} + \frac{C^2}{J + \mathrm{c}} = 1\,,$$

where J will serve us equally well as G; but here we have

$$\mathrm{a} + \mathrm{b} + \mathrm{c} = A^2 + B^2 + C^2,$$

and the developed equation in J will be

$$J^3 + [\mathrm{a}A^2 + \mathrm{b}B^2 + \mathrm{c}C^2 - \mathrm{a}^2 - \mathrm{b}^2 - \mathrm{c}^2 - \mathrm{ab} - \mathrm{bc} - \mathrm{ca}]\, J$$
$$- [\mathrm{bc}A^2 + \mathrm{ca}B^2 + \mathrm{ab}C^2 - \mathrm{abc}] = 0\,,$$

By the elimination of c and C and the partial reintroduction of a^2 and b^2, the shorter form is obtained,

$$J^3 - [\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]\, J + \mathrm{a}\,(b^2B^2 - \mathrm{ab}) + \mathrm{b}\,(a^2A^2 - \mathrm{ab}) = 0\,.$$

Employing the well-known trigonometric process for the solution of the cubic having all its roots real, we derive ψ (between the limits $\pm\, 90°$) from

$$\sin 3\psi = \frac{\sqrt{27}}{2}\, \frac{\mathrm{a}\,(b^2B^2 - \mathrm{ab}) + \mathrm{b}\,(a^2A^2 - \mathrm{ab})}{[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{3}{2}}}\,.$$

The roots of the cubic are then

$$J_x = -\frac{2}{\sqrt{3}}\,[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{1}{2}} \sin (60° + \psi),$$

$$J_y = \frac{2}{\sqrt{3}}\,[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{1}{2}} \sin \psi,$$

$$J_z = \frac{2}{\sqrt{3}}\,[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{1}{2}} \sin (60° - \psi).$$

To render these equations simpler, we will put

$$\mathrm{a}+\mathrm{b}=2n\cos\nu,\quad \sqrt{3}\,(\mathrm{a}-\mathrm{b})=2n\sin\nu,\quad a^2A^2+b^2B^2-2\,\mathrm{ab}=n^2\mu\cos\xi,$$

$$\frac{1}{\sqrt{3}}\left(a^2A^2-b^2B^2\right)=n^2\mu\sin\xi.$$

With these modifications we have

$$\sin 3\psi=\frac{\sqrt{27}}{2}\,\frac{\mu\cos(\nu+\xi)}{(1-\mu\cos\xi)^{\frac{3}{2}}},$$

$$J_x=-\frac{2}{\sqrt{3}}\,n\,(1-\mu\cos\xi)^{\frac{1}{2}}\sin(60^\circ+\psi),$$

$$J_y=\frac{2}{\sqrt{3}}\,n\,(1-\mu\cos\xi)^{\frac{1}{2}}\sin\psi,$$

$$J_z=\frac{2}{\sqrt{3}}\,n\,(1-\mu\cos\xi\,\sin)^{\frac{1}{2}}(60-\psi).$$

The modulus of the elliptic integrals involved, k, will be given by the equation

$$k^2=\sin^2\theta=\frac{J_y-J_x}{J_z-J_x}=\frac{\cos(60^\circ-\psi)}{\cos\psi.},$$

and the quantity we have designated by m, by the formula

$$m^2=J_z-J_x=2n\,(1-\mu\cos\xi)\,\tfrac{1}{2}\cos\psi.$$

The two elliptic integrals $M\sin^2\varkappa$, $M\cos^2 x$ are functions of ψ; consequently, they may be tabulated with the argument ψ or any function of ψ as, for instance, with $\sin 3\psi$, that is, with the absolute discriminant of the cubic. We have $\cos^2\theta=\dfrac{\sin(30^0-\psi)}{\cos\psi}$, and, if we put x^4 for the second member of this, we have

$$\left[\frac{\sin\dfrac{\theta}{2}}{1+\sqrt{\cos\theta}}\right]^2=\frac{1}{2}\,\frac{1-x}{1+x}=\frac{q+q^9+q^{25}+\ldots.}{1+2\,(q^4+q^{16}+q^{36}+\;\ldots.)}.$$

The value of the nome q can be derived from the infinite series (three terms suffice except in very unusual cases),

$$q=\frac{1}{2}\,\frac{1-x}{1+x}+\frac{1}{16}\left(\frac{1-x}{1+x}\right)^5+\frac{15}{512}\left(\frac{1-x}{1+x}\right)^9+\ldots.$$

As G_x, G_y, G_z enter into many of the preceding formulas, it may be useful to note that

$$\begin{aligned}
G_x \quad &= J_x + \mathrm{c}, & G_y \quad &= J_y + \mathrm{c}, & G_z \quad &= J_z + \mathrm{c}, \\
G_x + a^2 &= J_x + \mathrm{a}, & G_y + a^2 &= J_y + \mathrm{a}, & G_z + a^2 &= J_z + \mathrm{a}, \\
G_x + b^2 &= J_x + \mathrm{b}, & G_y + b^2 &= J_y + \mathrm{b}, & G_z + b^2 &= J_z + \mathrm{b}.
\end{aligned}$$

Variation of the Elements of a Planet through Perturbation.

The elements we select for use are defined thus: x, y, z, denoting the rectangular coordinates of a planet referred to the Sun as origin, adopt the elements c_x, c_y, c_z, f_x, f_y, f_z such that

$$c_x = \frac{ydz - zdy}{dt}, \qquad c_y = \frac{zdx - xdz}{dt}, \qquad c_z = \frac{xdy - ydx}{dt},$$

$$f_x = \frac{c_z dy - c_y dz}{dt} - \frac{\mu x}{r}, \quad f_y = \frac{c_x dz - c_z dx}{dt} - \frac{\mu y}{r}, \quad f_z = \frac{c_y dx - c_x dy}{dt} - \frac{\mu z}{r},$$

where we note that μ is the sum of the masses of the Sun and planet, and r is the radius vector of the latter. These six elements are not independent but satisfy the relation

$$c_x f_x + c_y f_y + c_z f_z = 0.$$

The additional element needed to complete the number six is the element everywhere attached by addition to the time. With these constants the two equations of the path of the planet in space are

$$c_x x + c_y y + c_z z = 0, \quad \mu r + f_x x + f_y y + f_z z = c_x^2 + c_y^2 + c_z^2 = k^2.$$

To understand the correlation of the different terms of these equations it must be borne in mind that μ is a constant of three dimensions in reference to the linear unit. Consequently, the c are of two dimensions and the f of three dimensions in reference to this unit.

The second equation belongs to a quadric surface not, in general, referred to its axes of symmetry. Removing the radical from it, it becomes

$$\begin{aligned}
(\mu^2 - f_x^2)\, x^2 + (\mu^2 - f_y^2)\, y^2 + (\mu^2 - f_z^2)\, z^2 - 2f_x f_y\, xy - 2f_y f_z\, yz - 2f_z f_x\, zx \\
+ 2k^2 f_x x + 2k^2 f_y y + 2k^2 f_z z - k^4 = 0.
\end{aligned}$$

The cubic, which must be solved in order that this may be referred to its axes of symmetry, is

$$\chi^3 - [3\mu^2 - (f_x^2 + f_y^2 + f_z^2)]\,\chi^2 + [(\mu^2 - f_x^2)(\mu^2 - f_y^2) + (\mu^2 - f_y^2)(\mu^2 - f_z^2)$$
$$+ (\mu^2 - f_z^2)(\mu^2 - f_x^2) - (f_x^2 f_y^2 + f_y^2 f_z^2 + f_z^2 f_x^2)]\,\chi - \mu^4\,[\mu^2 - (f_x^2 + f_y^2 + f_z^2)] = 0.$$

But, if e denotes the eccentricity, we have $f_x^2 + f_y^2 + f_z^2 = \mu^2 e^2$; thus the preceding equation reduces to

$$\chi^3 - \mu^2(3 - e^2)\,\chi^2 + \mu^4(3 - 2e^2)\,\chi - \mu^6(1 - e^2) = 0.$$

Put $\chi = \mu^2 x$, then

$$x^3 - (3 - e^2)\,x^2 + (3 - 2e^2)\,x - (1 - e^2) = 0,$$

or

$$(x - 1)^3 + e^2(x - 1)^2 = 0.$$

The three roots are $x = 1$, $x = 1$, $x = 1 - e^2$. The second equation of the orbit of the planet is, therefore, a quadric of revolution about the major axis, in the present investigation an ellipsoid, as we suppose $e^2 < 1$. The expression $c_x^2 + c_y^2 + c_z^2$ is invariant, as also is $f_x^2 + f_y^2 + f_z^2$, both being independent of the orientation of the axes of coordinates. The system of principal axes will be arrived at if we suppose $f_y = 0, f_z = 0$, which imply also $c_x = 0$, as is shown by the relation $c_x f_x + c_y f_y + c_z f_z = 0$, unless we have $f_x = 0$, when, the quadric being a sphere, all systems of axes are principal. The equations of the orbit then take the form

$$c_y y + c_z z = 0, \quad \mu r + \mu e x = k^2,$$

where $k^2 = c_y^2 + c_z^2$. The radical removed, the second equation becomes

$$\mu^2(1 - e^2)\,x^2 + \mu^2 y^2 + \mu^2 z^2 + 2k^2\mu e x - k^4 = 0.$$

But, a being the semi-axis major, $k^2 = \mu a(1 - e^2)$, therefore, the preceding equation takes the form

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{a^2(1 - e^2)} + 2\,\frac{e}{a}\,x - (1 - e^2) = 0,$$

or

$$\frac{(x + ae)^2}{a^2} + \frac{y^2 + z^2}{a^2(1 - e^2)} = 1.$$

The adoption of the attracted planet instead of the centre of the Sun as the origin of coordinates renders it necessary, while employing the elements of the planet, to substitute $-x_0$, $-y_0$, $-z_0$ for x, y, z. Thus, with this notation, the equations of the Sun's path in space are

$$c_x x_0 + c_y y_0 + c_z z_0 = 0, \quad \mu r_0 - f_x x_0 - f_y y_0 - f_z z_0 = k^2.$$

The values of the differentials with respect to the time of the constants c and f must now be stated. Denoting the mass of the attracting planet by m', the perturbative function R for secular perturbations is $\frac{m'}{\rho}$, and we have

$$\frac{dc_x}{dt} = y_0 \frac{\partial R}{\partial z} - z_0 \frac{\partial R}{\partial y} = \frac{m'}{\rho^3}(z_0 y - y_0 z),$$

$$\frac{dc_y}{dt} = z_0 \frac{\partial R}{\partial x} - x_0 \frac{\partial R}{\partial z} = \frac{m'}{\rho^3}(x_0 z - z_0 x),$$

$$\frac{dc_z}{dt} = x_0 \frac{\partial R}{\partial y} - y_0 \frac{\partial R}{\partial x} = \frac{m'}{\rho^3}(y_0 x - x_0 y),$$

$$\frac{df_x}{dt} = \frac{dz_0}{dt}\frac{dc_y}{dt} - \frac{dy_0}{dt}\frac{dc_z}{dt} + \frac{m'}{\rho^3}(c_z y - c_y z),$$

$$\frac{df_y}{dt} = \frac{dx_0}{dt}\frac{dc_z}{dt} - \frac{dz_0}{dt}\frac{dc_x}{dt} + \frac{m'}{\rho^3}(c_x z - c_z x),$$

$$\frac{df_z}{dt} = \frac{dy_0}{dt}\frac{dc_x}{dt} - \frac{dx_0}{dt}\frac{dc_y}{dt} + \frac{m'}{\rho^3}(c_y x - c_x y). \; *$$

There is still one element to be added to the preceding to complete the number of six independent constants; this is the mean longitude at epoch. The well-known equation for its variation shows that a portion is immediately derivable from the motion of the perihelion, and another from that of the node; what remains (calling the element l) is given by the equation

$$\frac{dl}{dt} = -2\frac{an}{\mu}\left[x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right] = 2an\frac{m'}{\mu}\left[x_0\frac{x}{\rho^3} + y_0\frac{y}{\rho^3} + z_0\frac{z}{\rho^3}\right],$$

where n denotes the mean motion of the attracted planet.

The system just given is the most general as respects the orientation of the axes of coordinates. Let us now specialize by taking, for the plane of xy, the plane of the orbit of the attracted planet. This makes $c_x = 0$, $c_y = 0$, $c_z = k$ and the equation between the c and the f shows that, in consequence, we have $f_z = 0$. Then our equations become

$$\frac{dc_x}{dt} = -\frac{m'}{\rho^3} y_0 z, \qquad \frac{df_x}{dt} = -\frac{dy_0}{dt}\frac{dk}{dt} + k\frac{m'}{\rho^3} y,$$

$$\frac{dc_y}{dt} = \frac{m'}{\rho^3} x_0 z, \qquad \frac{df_y}{dt} = \frac{dx_0}{dt}\frac{dk}{dt} - k\frac{m'}{\rho^3} x,$$

$$\frac{dk}{dt} = \frac{m'}{\rho^3}(y_0 x - x_0 y), \qquad \frac{df_z}{dt} = \frac{dy_0}{dt}\frac{dc_x}{dt} - \frac{dx_0}{dt}\frac{dc_y}{dt}.$$

* For these formulas consult Laplace, Mécanique Celeste, Tom. I, Liv. II, Art. 64.

But our specialization, applied to the definitions of the f, gives

$$-k\frac{dx_0}{dt}=\mu\frac{y_0}{r_0}-f_y,\quad k\frac{dy_0}{dt}=\mu\frac{x_0}{r_0}-f_x.$$

These values enable us to eliminate the differentials of x_0, y_0 from the preceding equations. By differentiating the equation between the c and f we get

$$f_x\frac{dc_x}{dt}+f_y\frac{dc_y}{dt}+k\frac{df_z}{dt}=0.$$

The substitution of preceding values in this renders it an identity, hence the equation for $\frac{df_z}{dt}$ is superfluous. Moreover, after use in substitution, the equation for $\frac{dk}{dt}$ is no longer needed, since the secular motion of the semi-axis a vanishes. Thus the equations to be employed are reduced to the five following:

$$\begin{aligned}
\frac{dc_x}{dt}&=-m'y_0\frac{z}{\rho^3},\\
\frac{dc_y}{dt}&=\quad m'x_0\frac{z}{\rho^3},\\
\frac{df_x}{dt}&=\frac{m'}{k}\left(f_x-\mu\frac{x_0}{r_0}\right)\left(y_0\frac{x}{\rho^3}-x_0\frac{y}{\rho^3}\right)+m'k\frac{y}{\rho^3},\\
\frac{df_y}{dt}&=\frac{m'}{k}\left(f_y-\mu\frac{y_0}{r_0}\right)\left(y_0\frac{x}{\rho^3}-x_0\frac{y}{\rho^3}\right)-m'k\frac{x}{\rho^3},\\
\frac{dl}{dt}&=2an\frac{m'}{\mu}\left(x_0\frac{x}{\rho^3}+y_0\frac{y}{\rho^3}\right).
\end{aligned}$$

If χ denote the longitude of the perihelion of the attracted planet, we have $f_x=\mu e\cos\chi$, $f_y=\mu e\sin\chi$; also, if i is the inclination and ☊ the longitude of the ascending node, $c_x=k\sin i\sin ☊$, $c_y=-k\sin i\cos ☊$; whence

$$\begin{aligned}
\frac{d(\sin i\sin ☊)}{dt}&=-\frac{m'}{k}y_0\frac{z}{\rho^3},\\
\frac{d(\sin i\cos ☊)}{dt}&=-\frac{m'}{k}x_0\frac{z}{\rho^3},\\
\frac{d(e\sin\chi)}{dt}&=\frac{m'}{k}\left(e\sin\chi-\frac{y_0}{r_0}\right)\left(y_0\frac{x}{\rho^3}-x_0\frac{y}{\rho^3}\right)-\frac{m'}{\mu}k\frac{x}{\rho^3},\\
\frac{d(e\cos\chi)}{dt}&=\frac{m'}{k}\left(e\cos\chi-\frac{x_0}{r_0}\right)\left(y_0\frac{x}{\rho^3}-x_0\frac{y}{\rho^3}\right)+\frac{m'}{\mu}k\frac{y}{\rho^3}.
\end{aligned}$$

The point of departure for longitudes is still undetermined; but, if this is taken to be the ascending node of the attracted planet's orbit on that of the attracting planet, we can employ the X''', Y''', Z''' we have previously defined. Thus, using brackets with subscript 0 to denote integration along the orbit of the attracting planet we have the equations

$$\left[\frac{d(\sin i \sin \Omega)}{dt}\right]_0 = -\frac{m'}{k} y_0 Z''',$$

$$\left[\frac{d(\sin i \cos \Omega)}{dt}\right]_0 = -\frac{m'}{k} x_0 Z''',$$

$$\left[\frac{d(e \sin \chi)}{dt}\right]_0 = \frac{m'}{k}\left(e \sin \chi - \frac{y_0}{r_0}\right)\left(y_0 X''' - x_0 Y'''\right) - \frac{m'}{\mu} k X''',$$

$$\left[\frac{d(e \cos \chi)}{dt}\right]_0 = \frac{m'}{k}\left(e \cos \chi - \frac{x_0}{r_0}\right)\left(y_0 X''' - x_0 Y'''\right) + \frac{m'}{\mu} k Y''',$$

$$\left[\frac{dl}{dt}\right]_0 = 2an \frac{m'}{\mu}\left(x_0 X''' + y_0 Y'''\right).$$

The integration round the orbit of the disturbed planet has still to be executed in order to arrive at the secular motion of the elements. For this we are confined to the use of mechanical quadratures. Here we may use either of the three anomalies, or any variable which will show the position of the planet on its orbit, as the independent variable.

WEST NYACK, *March* 13, 1901.

Representation of Linear Groups as Transitive Substitution Groups.

By Leonard Eugene Dickson.

Introduction.

One of the advantages of the study of groups of congruences and of groups of linear substitutions in a general Galois field is the ability to deal with groups of high order as well as with infinite systems of groups by means of analytic formulæ involving a small number of variables. The study of finite groups defined analytically has led to such distinctive methods and the results have been given such a degree of generality that there appears to be some justification for the attitude of many specialists in the theory of substitution groups towards the analytic theory. It is hoped that the present investigation will be a first step in the direction of a closer union of these branches of group theory.

After giving in §1 a proof of the known theorem on the representation of certain quotient-groups derived from the *general* linear group as doubly transitive substitution groups, and an outline in §3 of the general method of the paper for representing the more important classes of *special* linear groups, I take up the orthogonal groups in §§4–16, the abelian linear group in §§17–20, and, finally, the hypoabelian groups in §§21–26. A similar investigation for the hyperorthogonal linear group* will form part of a paper to be presented to the Annalen. In the main, the paper is complete within itself, but occasional aid is drawn chiefly from the memoir† in which I made a study of all linear groups defined by a quadratic invariant, all of these groups being now represented as transitive substitution groups.

The group of orthogonal substitutions of determinant unity in the $GF[p^n]$ on an odd number m of variables may be represented as a transitive substitution

* Mathematische Annalen, vol. LII, pp. 561–581. † American Journal, vol. XXI, pp. 193–256.

group on $(p^{n(m-1)}-1)/(p^n-1)$ letters. Except for $p^n=3$, $m \gtreqless 5$, this is the minimum number of letters for the methods of representation here studied. For $m=3$, the above number is p^n+1, in accord with the known* isomorphism of a subgroup of index 2 of the ternary orthogonal group of substitutions of determinant unity with the linear fractional group, each in the $GF[p^n]$. Aside from the cases $p^n=5, 7, 9, 11$, the latter group cannot be represented on fewer than p^n+1 letters (Moore, Wiman; and Galois, Gierster, for $n=1$).

The minimum number of letters necessary for the representation (by the method under investigation) of the simple groups derived from the orthogonal group on an even number m of variables is given in §§10, 11. For the quaternary second orthogonal quotient-group $Q^{(\nu)}_{4,p^n}$, this number is $p^{2n}+1$, in accord with the isomorphism† of its subgroup of index 2 with the linear fractional group in the $GF[p^{2n}]$.

These results serve on the one hand as a check and on the other hand as an indication that the method gives the minimum number of letters when p^n exceeds a certain low value.

As a special result of the investigation of the $2m$-ary hypoabelian groups (the case $n=1$), we note that the first and second hypoabelian groups may be represented as transitive substitution groups on

$$2^{2m-1}-2^{m-1}, \quad 2^{2m-1}-2^{m-1}-1$$

letters respectively (minimum values for the method here employed). This result agrees with the known isomorphism of the first hypoabelian group and the Steiner substitution group.

Finally, I would call special attention to the important property of various linear groups explained in §2.

1. A linear homogeneous substitution on m variables,

$$A: \quad \xi_i' = \sum_{j=1}^{m} \alpha_{ij}\xi_j, \qquad (i=1, 2, \ldots, m),$$

with coefficients in the Galois field of order p^n, the $GF[p^n]$, permutes amongst themselves the linear functions

$$\lambda_1\xi_1+\lambda_2\xi_2+\ldots+\lambda_m\xi_m, \qquad (\lambda_i \text{ in the } GF[p^n]).$$

* American Journal, l. c., p. 219.

† Ibid., pp. 249-255. On p. 255, F_{1,p^n}, F_{1,p_n} should read $F_{1,p^{2n}}$.

Excluding the case $\lambda_1 = \lambda_2 = \ldots\ldots = \lambda_m = 0$, these functions may be combined in sets of $p^n - 1$, those of any set being of the form

$$\mu\lambda_1\xi_1 + \mu\lambda_2\xi_2 + \ldots\ldots + \mu\lambda_m\xi_m,$$

where μ runs through the series of marks $\neq 0$ of the $GF[p^n]$. Such a set will be designated by the symbol $\{\lambda_1\xi_1 + \ldots\ldots + \lambda_m\xi_m\}$, so that there are $(p^{nm} - 1)/(p^n - 1)$ distinct symbols. A substitution A which leaves every symbol unaltered evidently has the form

$$R: \quad \xi_i' = \rho\xi_i, \qquad (i = 1, 2, \ldots\ldots, m).$$

A substitution R is commutative with every linear substitution A, so that the group of all linear substitutions A has an invariant subgroup formed of the substitutions R. The quotient-group Γ is holoedrically isomorphic with the substitution group G on the above symbols.

Moreover, G is doubly transitive. In proof, it is only necessary to show that, if two distinct symbols

$$\{\lambda_1\xi_1 + \lambda_2\xi_2 + \ldots\ldots + \lambda_m\xi_m\}, \quad \{\rho_1\xi_1 + \rho_2\xi_2 + \ldots\ldots + \rho_m\xi_m\}$$

are given, such, therefore, that the ratios $\lambda_1 : \rho_1, \ldots\ldots, \lambda_m : \rho_m$ are not all equal, there exists a linear substitution which replaces the symbols $\{\xi_1\}$ and $\{\xi_2\}$ respectively by the first and the second given symbols. If α_{ij} be chosen, as may be done, so that the determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \ldots\ldots & \lambda_m \\ \rho_1 & \rho_2 & \ldots\ldots & \rho_m \\ \ldots & \ldots & \ldots\ldots & \ldots \\ \alpha_{i1} & \alpha_{i2} & \ldots\ldots & \alpha_{im} \\ \ldots & \ldots & \ldots\ldots & \ldots \end{vmatrix} \neq 0,$$

the required substitution may be taken to be

$$\xi_1' = \sum_{j=1}^{m} \lambda_j\xi_j, \quad \xi_2' = \sum_{j=1}^{m} \rho_j\xi_j, \quad \xi_i' = \sum_{j=1}^{m} \alpha_{ij}\xi_j, \qquad (i = 3, \ldots\ldots, m).$$

The quotient-group Γ may be represented as a doubly transitive substitution group on $(p^{nm} - 1)/(p^n - 1)$ letters.

Of more importance than Γ is the quotient-group $LF(m, p^n)$ of the group of all linear substitutions A of determinant unity by the invariant subgroup formed of the substitutions R of determinant unity. The notation $LF(m, p^n)$ is derived

from the linear fractional form in which its operators may be exhibited. Except for $p^n = 2$, $m = 2$; $p^n = 3$, $m = 3$, the group $LF(m, p^n)$ is simple.*

The above proof may be extended to the group $LF(m, p^n)$; only the ratios of the λ_i being essential, the determinant employed may be supposed equal to unity. Hence the theorem:

The linear fractional group $LF(m, p^n)$ may be represented as a doubly transitive substitution group on $(p^{nm} - 1)/(p^n - 1)$ letters.

If this method of procedure be employed in the case of a linear group possessing an invariant $\phi(\xi_1, \xi_2, \ldots, \xi_m)$, it evidently leads to an intransitive substitution group, and the number of letters is unnecessarily large. The present paper presents a general method of representing such groups as transitive permutation groups and determines which representation involves the fewest letters

2. Frequent application is made of a fundamental property possessed by various linear groups. Of the conditions upon the coefficients of the general substitution of a given linear group, let R_1 denote those involving only the coefficients of the first row of the matrix of coefficients, $R_{1,2}$ those involving only the coefficients of the first and second rows, etc. When the linear group contains a substitution in which the coefficients of the first row are *arbitrary* marks of the field which satisfy conditions R_1, a substitution in which the coefficients of the first and second rows are arbitrary marks satisfying conditions $R_{1,2}$, etc., the group will be said to possess *successive generality*.

The orthogonal, the abelian, hypoabelian and the hyperorthogonal groups possess successive generality. Certain linear groups†, the conditions upon whose coefficients are of degree > 2, do not possess this property, at least when not reduced in form [for example, if we use the conditions given by A^{-1}, §3].

3. In order that a linear substitution A shall leave formally invariant a function $\phi(\xi_1, \xi_2, \ldots, \xi_m)$, certain conditions upon the coefficients must be satisfied. Thus $\phi(\xi_1', \xi_2', \ldots, \xi_m') \equiv \phi$ requires

$$\phi(\alpha_{11}, \alpha_{21}, \ldots, \alpha_{m1}) = \phi(1, 0, \ldots, 0),$$
$$\phi(\alpha_{12}, \alpha_{22}, \alpha_{32}, \ldots, \alpha_{m2}) = \phi(0, 1, 0, \ldots, 0).$$

But conditions of this nature involve the coefficients of one or more columns of the

* Annals of Mathematics, vol. XI, pp. 161-183; University of Chicago Record, Aug., 1896.
† Quarterly Journal, July, 1899; Proc. Lond. Math. Soc., vol. XXX, p. 200.

matrix for A. To obtain the conditions involving only the coefficients of one or more rows of the matrix for A, it usually suffices to require that ϕ be invariant under A^{-1}, the inverse of the general substitution of the group. The set of conditions obtained by means of A^{-1} must of course be equivalent to the set obtained from A, but are usually in a more convenient form.

For definiteness of expression, suppose there is a single condition $\psi(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m}) = c$ involving only the coefficients of the first row of the matrix of the general substitution A of the group. The substitution A replaces $\lambda_1 \xi_1 + \lambda_2 \xi_2 + \ldots + \lambda_m \xi_m$ by the function

$$\sum_{j=1}^{m} \gamma_j \xi_j, \quad \gamma_j \equiv \sum_{i=1}^{m} \alpha_{ij} \lambda_i,$$

the matrix of the coefficients of the λ^i in the γ_j being the transposed of the matrix A, viz.,

$$\begin{array}{r|llll} & \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \hline \gamma_1 = & \alpha_{11} & \alpha_{21} & \ldots & \alpha_{m1} \\ \gamma_2 = & \alpha_{12} & \alpha_{22} & \ldots & \alpha_{m2} \\ & \ldots & \ldots & \ldots & \ldots \\ \gamma_m = & \alpha_{1m} & \alpha_{2m} & \ldots & \alpha_{mm}. \end{array}$$

Hence, if the transposed of the matrix of the general substitution of a linear group always defines a substitution of the group, the functions $\lambda_1 \xi_1 + \ldots + \lambda_m \xi_m$, where $\phi(\lambda_1, \lambda_2, \ldots, \lambda_m)$ has a constant value, are permuted amongst themselves. But this condition on the transposed matrix may not be satisfied, as, for example, when the group is the second orthogonal group in the $GF[p^n]$, $p^n = 4l + 1$ [See §4, end].

If $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a set of marks, necessarily satisfying

$$(R_1) \qquad \psi(\lambda_1, \lambda_2, \ldots, \lambda_m) = c,$$

such that the group contains at least one substitution T replacing ξ_1 by $\sum_{i=1}^{m} \lambda_i \xi_i$, then the product ST belongs to the group, and hence replaces ξ_1 by a function $\Sigma \gamma_j \xi_j$ for which

$$\psi(\gamma_1, \gamma_2, \ldots, \gamma_m) = c.$$

We thus obtain a set of functions which are permuted by the group. For the groups studied in this paper, the above assumption will be satisfied for an arbitrary set of solutions $\lambda_1, \ldots, \lambda_m$ of (R_1).

The following generalization is immediate. Its statement is, however, limited to the extensive classes of linear groups possessing successive generality (§2). Employing the notations R_1, $R_{1,2}$,.. of §2, we obtain a first representation of the group as a substitution group by considering the linear functions $\lambda_1\xi_1 + \lambda_2\xi_2 + \ldots\ldots + \lambda_m\xi_m$, where $\lambda_1, \ldots\ldots, \lambda_m$ are any marks satisfying the contions R_1; a second representation by means of a pair of linear functions $\Sigma\lambda_{1i}\,\xi_i$ and $\Sigma\lambda_{2i}\,\xi_i$, where λ_{1i}, λ_{2i} are any marks which satisfy the conditions $R_{1,2}$; a third representation by means of three linear functions with coefficients satisfying the conditions $R_{1,2,3}$; finally, a representation by means of m linear functions whose coefficients are the most general set of m^2 marks which satisfy the conditions $R_{1,2,\ldots\ldots,m}$ imposed upon the general substitution of the group. Instead of the linear functions, we may, in each case, use a positional symbol, the elements of each row of which are the coefficients of the successive linear functions, thus:

$$[\lambda_1, \lambda_2, \ldots\ldots, \lambda_m], \quad \begin{bmatrix} \lambda_{11}, \lambda_{12}, \ldots\ldots, \lambda_{1m} \\ \lambda_{21}, \lambda_{22}, \ldots\ldots, \lambda_{2m} \end{bmatrix}, \ldots\ldots, \begin{bmatrix} \lambda_{11}, \lambda_{12}, \ldots\ldots, \lambda_{1m} \\ \lambda_{21}, \lambda_{22}, \ldots\ldots, \lambda_{2m} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_{m1}, \lambda_{m2}, \ldots\ldots, \lambda_{mm} \end{bmatrix}.$$

In the last case, the group is represented as a regular substitution group; indeed, the symbol represents the matrix of the general substitution of the group, so that the process, for this case, is identical with that used by Dyck.*

The First and Second Orthogonal Groups, §§4–16.

4. Every group of linear homogeneous substitutions on m variables with coefficients in the $GF[p^n]$, $p > 2$, which is defined by a quadratic invariant of non-vanishing determinant, can be transformed by a linear homogeneous substitution belonging to the field into one of the two groups.

1°. The first orthogonal group $O^{(1)}_{m,p}$ with the invariant

$$\xi_1^2 + \xi_2^2 + \ldots\ldots + \xi_{m-1}^2 + \xi_m^2.$$

2°. The second orthogonal group $O^{(\nu)}_{m,p^n}$ with the invariant

$$\xi_1^2 + \xi_2^2 + \ldots\ldots + \xi_{m-1}^2 + \nu\xi_m^2,$$

* Mathematische Annalen, vol. XX (1882), p. 30.

where ν is a particular not-square in the $GF[p^n]$. For m odd, the second group is conjugate with the first.*

Treating together the two groups, let $O^{(\mu)}_{m, p^n}$ denote the orthogonal group defined by the invariant

$$\xi_1^2 + \xi_2^2 + \cdots + \xi_{m-1}^2 + \mu\xi_m^2.$$

The conditions upon the coefficients of a substitution

$$A: \quad \xi_i' = \sum_{j=1}^{m} \alpha_{ij}\,\xi_j \qquad (i = 1, \ldots, m)$$

belonging to the group are known to be the following:

$$\sum_{i=1}^{m-1} \alpha_{ij}^2 + \mu\alpha_{mj}^2 = \begin{cases} 1, & (j = 1, 2, \ldots, m-1), \\ \mu, & (j = m), \end{cases} \tag{1}$$

$$\sum_{i=1}^{m-1} \alpha_{ij}\,\alpha_{ik} + \mu\alpha_{mj}\,\alpha_{mk} = 0, \quad (j, k = 1, \ldots, m;\ j \neq k). \tag{2}$$

The inverse of A is, therefore,

$$A^{-1}: \quad \begin{cases} \xi_i' = \sum_{j=1}^{m-1} \alpha_{ji}\,\xi_j + \mu\alpha_{mi}\xi_m, & (i = 1, 2, \ldots, m-1). \\ \xi_m' = \dfrac{1}{\mu}\sum_{j=1}^{m-1} \alpha_{jm}\,\xi_j + \alpha_{mm}\,\xi_m. \end{cases}$$

Writing the relations (1) and (2) for the inverse A^{-1}, we have

$$\sum_{i=1}^{m-1} \alpha_{ji}^2 + \frac{1}{\mu}\,\alpha_{jm}^2 = \begin{cases} 1\ , & (j = 1, 2, \ldots, m-1), \\ 1/\mu, & (j = m). \end{cases} \tag{1'}$$

$$\sum_{i=1}^{m-1} \alpha_{ji}\,\alpha_{ki} + \frac{1}{\mu}\,\alpha_{jm}\,\alpha_{km} = 0, \quad (j, k = 1, \ldots, m;\ j \neq k). \tag{2'}$$

It follows that the transposed of the matrix of A will likewise belong to the group if, and only if, either $\mu^2 = 1$ or else

$$\alpha_{jm} = 0, \quad \alpha_{mj} = 0, \quad \alpha_{mm}^2 = 1, \quad (j = 1, 2, \ldots, m-1).$$

In the latter case the substitution leaves $\xi_1^2 + \cdots + \xi_{m-1}^2$ and ξ_m^2 each invariant.

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* Dickson, American Journal, vol. XXI, pp. 193-256. This paper will be referred to as A. J., with the specific pages mentioned. Certain changes of notation, however, will be found desirable for the present use.

5. THEOREM.—*The orthogonal groups possess successive generality.*

Given a set of marks α'_{ij} of the $GF[p^n]$ such that

$$\sum_{i=1}^{m-1} \alpha'^2_{ji} + \frac{1}{\mu}\alpha'^2_{jm} = 1, \quad \sum_{i=1}^{m-1} \alpha'_{ji}\alpha'_{ki} + \frac{1}{\mu}\alpha'_{jm}\alpha'_{km} = 0, \tag{3}$$

for $j, k = 1, 2, \ldots, r, j \neq k$, where r is a given positive integer* $< m$, it is to be shown that the group $O^{(\mu)}_{m,p^n}$ contains a substitution S which replaces ξ_j by $\sum_{i=1}^{m} \alpha'_{ji}\xi_i$ for $j = 1, 2, \ldots, r$.

The theorem may be established by induction. Assuming it to be true for the case $r-1$, it will be proved true for the case r. The theorem is true for $r = 1$ (*A. J.*, pp. 199–207; see §6 below). By the hypothesis, the group contains a substitution A' of the form A (§4) in which

$$\alpha_{ij} \equiv \alpha'_{ij} \text{ for } i = 1, 2, \ldots, r-1; j = 1, \ldots, m. \tag{4}$$

By (3), $\alpha'_{r1}, \alpha'_{r2}, \ldots, \alpha'_{rm}$ do not all vanish; moreover, the determinants of the matrix

$$\begin{vmatrix} \alpha'_{11} & \alpha'_{12} & \ldots & \alpha'_{1m} \\ \alpha'_{21} & \alpha'_{22} & \ldots & \alpha'_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha'_{r1} & \alpha'_{r2} & \ldots & \alpha'_{rm} \end{vmatrix}$$

are not all zero. Hence there exists an m-ary linear substitution S_1 in the $GF[p^n]$ of non-vanishing determinant which, like the required substitution S, replaces ξ_j by $\sum_{i=1}^{m} \alpha'_{ji}\xi_i$ for $j = 1, 2, \ldots, r$. The product $A'^{-1}S_1 \equiv R$ leaves $\xi_1, \xi_2, \ldots, \xi_{r-1}$ each fixed in view of (4) and replaces ξ_r by

$$R_1\xi_1 + R_2\xi_2 + \ldots + R_m\xi_m,$$

where, for $i = 1, 2, \ldots, m-1$,

$$R_i \equiv \sum_{j=1}^{m-1} \alpha'_{rj}\alpha_{ij} + \frac{1}{\mu}\alpha'_{rm}\alpha_{im}, \quad R_m \equiv \mu\sum_{j=1}^{m-1} \alpha'_{rj}\alpha_{mj} + \alpha'_{rm}\alpha_{mm}. \tag{5}$$

In view of (3) and (4), $R_1 = R_2 = \ldots = R_{r-1} = 0$, while

$$\sum_{i=1}^{m-1} R_i^2 + \frac{1}{\mu}R_m^2 = \sum_{j=1}^{m-1} \alpha'^2_{rj}\left\{\sum_{i=1}^{m-1} \alpha^2_{ij} + \mu\alpha^2_{mj}\right\} + \frac{1}{\mu^2}\alpha'^2_{rm}\left\{\sum_{i=1}^{m-1} \alpha^2_{im} + \mu\alpha^2_{mm}\right\}$$

$$+ 2\sum_{j<k}^{1,\ldots,m-1} \alpha'_{rj}\alpha'_{rk}\left\{\sum_{i=1}^{m-1} \alpha_{ij}\alpha_{ik} + \mu\alpha_{mj}\alpha_{mk}\right\} + \frac{2}{\mu}\sum_{j=1}^{m-1} \alpha'_{rj}\alpha'_{rm}\left\{\sum_{i=1}^{m-1} \alpha_{ij}\alpha_{im} + \mu\alpha_{mj}\alpha_{mm}\right\}$$

$$= \sum_{j=1}^{m-1} \alpha'^2_{rj} + \frac{1}{\mu}\alpha'^2_{rm} = 1,$$

* For $r = m$, the theorem is true by the definition of the groups.

upon applying relations (1) and (2) and afterwards (3). Inversely, the substitution $S_1 \equiv A'R$ replaces ξ_j by the linear function $\sum_{i=1}^{m} \alpha'_{ji}\, \xi_i$ for $j = 1, 2, \ldots, r$. Hence if we construct a substitution R belonging to the orthogonal group such that R leaves fixed $\xi_1, \xi_2, \ldots, \xi_{r-1}$ and replaces ξ_r by the function

$$\sum_{i=r}^{m} R_i\, \xi_i, \quad \sum_{i=r}^{m-1} R_i^2 + \frac{1}{\mu} R_m^2 = 1,$$

the product $A'R$ may be taken as the required orthogonal substitution S.

As shown above, the quantities R_i may be defined by (5). But such a substitution R exists in the group $O^{(\mu)}_{m-r+1}$ and *à fortiori* in the group $O^{(\mu)}_{m,p^n}$ in view of the theorem above quoted (the present theorem for the case $r = 1$).

In a similar manner it may be shown that the group of orthogonal substitutions of determinant $+1$ possesses successive generality. Indeed, the theorem is true for $r = 1$, and the necessary modifications in the above proof are evident.*

6. There is a second method of proof which establishes the theorem for every r including $r = 1$. The theorem is first proved for the case $r = m - 1$ as follows. Given a set of marks

$$\alpha'_{j1}, \alpha'_{j2}, \ldots, \alpha'_{jm}, \qquad (j = 1, 2, \ldots, m-1), \quad (6,$$

satisfying relations (3) for $r = m - 1$ and therefore for $j, k = 1, 2, \ldots, m-1$; $k \neq j$, there exists a single set of marks $\alpha'_{m1}, \alpha'_{m2}, \ldots, \alpha'_{mm}$, such that the substitution

$$S:\quad \xi'_j = \sum_{i=1}^{m} \alpha'_{ji}\, \xi_i \qquad (j = 1, 2, \ldots, m)$$

has determinant unity and belongs to the group, $O^{(\mu)}_{m,p^n}$, viz.,

$$\alpha'_{m1} = (-1)^{m-1} \begin{vmatrix} \alpha'_{12} & \alpha'_{13} & \cdots & \alpha'_{1m} \\ \alpha'_{22} & \alpha'_{23} & \cdots & \alpha'_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha'_{m-12} & \alpha'_{m-13} & \cdots & \alpha'_{m-1m} \end{vmatrix}, \ldots, \alpha'_{mm} = \begin{vmatrix} \alpha'_{11} & \alpha'_{12} & \cdots & \alpha'_{1m-1} \\ \alpha'_{21} & \alpha'_{22} & \cdots & \alpha'_{2m-1} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha'_{m-11} & \alpha'_{m-12} & \cdots & \alpha'_{m-1m-1} \end{vmatrix}.$$

Since these expressions are the first minors (with proper sign prefixed) of α'_{mi} in the determinant $|\alpha'_{ji}|$, this result agrees with that giving the form of the inverse A^{-1} of the general orthogonal substitution A.

* In regard to the special rôle played by the index ξ_m, compare § 23.

Taking the negatives of these expressions as the values of the unknown $\alpha'_{m1}, \ldots, \alpha'_{mm}$, we obtain the unique orthogonal substitution S of determinant -1 having the prescribed coefficients (6) in the first $m-1$ rows of the matrix.

The proof of the general theorem of §5 would now proceed from the case $r=m-1$ to the case $r=m-2$, etc. For the first step, $\alpha'_{m-11}, \ldots, \alpha'_{m-1m}$ are to be determined so as to satisfy a quadratic relation and $m-2$ linear relations, the latter involving the (given) coefficients of the preceding $m-2$ rows of the matrix.

The method will be illustrated by the important case $m=3$. Given any set of marks $\alpha_{11}, \alpha_{12}, \alpha_{13}$ such that

$$\alpha_{11}^2+\alpha_{12}^2+\frac{1}{\mu}\alpha_{13}^2=1, \tag{7}$$

we are to determine marks $\alpha_{21}, \alpha_{22}, \alpha_{23}$ such that

$$\alpha_{21}^2+\alpha_{22}^2+\frac{1}{\mu}\alpha_{23}^2=1,\quad \alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}+\frac{1}{\mu}\alpha_{13}\alpha_{23}=0. \tag{8}$$

If any $\alpha_{1i}=0$, a solution is evident. Thus, for $\alpha_{12}=0$, we take $\alpha_{21}=\alpha_{23}=0$, $\alpha_{22}=1$. If $\alpha_{11}=0$, take $\alpha_{21}=1$, $\alpha_{22}=\alpha_{23}=0$. If $\alpha_{13}=0$, take

$$\alpha_{23}=0,\quad \alpha_{21}=-\alpha_{12},\quad \alpha_{22}=\alpha_{11}.$$

If every $\alpha_{1i}\neq 0$, we eliminate α_{21} from (8) and obtain

$$\left(\alpha_{11}^2+\alpha_{12}^2\right)\alpha_{22}^2+\left(\alpha_{11}^2+\frac{1}{\mu}\alpha_{13}^2\right)\alpha_{23}^2/\mu+\frac{2}{\mu}\alpha_{12}\alpha_{13}\alpha_{22}\alpha_{23}=\alpha_{11}^2. \tag{9}$$

If both $\alpha_{11}^2+\alpha_{12}^2$ and $\alpha_{11}^2+\frac{1}{\mu}\alpha_{13}^2$ vanish, the equation determines $\alpha_{22}\alpha_{23}$, so that α_{23} may be chosen as an arbitrary mark $\neq 0$, α_{21} being determined by the second condition (8). In the contrary case, we may take $\alpha_{11}^2+\alpha_{12}^2\neq 0$.* The equation (9) is therefore equivalent to the equation

$$\left\{\left(\alpha_{11}^2+\alpha_{12}^2\right)\alpha_{22}+\frac{1}{\mu}\alpha_{12}\alpha_{13}\alpha_{23}\right\}^2 +\frac{1}{\mu}\alpha_{11}^2\left(\alpha_{11}^2+\alpha_{12}^2+\frac{1}{\mu}\alpha_{13}^2\right)\alpha_{23}^2=\alpha_{11}^2\left(\alpha_{11}^2+\alpha_{12}^2\right).$$

In view of (7), the coefficient of α_{23}^2 is not zero. By the theorem quoted in §9,

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* For if $\alpha_{11}^2+\alpha_{12}^2=0$, $\alpha_{11}^2+\frac{1}{\mu}\alpha_{13}^2\neq 0$, then, by (7) μ is a square and therefore unity, so that $\alpha_{11}^2+\alpha_{13}^2\neq 0$ and the equations are symmetrical in $\alpha_{j1}, \alpha_{j2}, \alpha_{j3}$.

the equation has solutions in the $GF[p^n]$,

$$\alpha_{23}, \quad \left(\alpha_{11}^2 + \alpha_{12}^2\right)\alpha_{22} + \frac{1}{\mu}\alpha_{12}\,\alpha_{13}\,\alpha_{23},$$

and therefore solutions α_{23}, α_{22} in the field. Hence the result:*

For any set of solutions in the $GF[p^n]$, $p \neq 2$, *of equation* (7), *there exists a substitution of determinant unity in the group* $O^{(\mu)}_{3,p^n}$ *which replaces* ξ_1 *by* $\alpha_{11}\xi_1 + \alpha_{12}\xi_2 + \alpha_{13}\xi_3$.

Employing this theorem, the corresponding theorem for m variables is readily proved by induction (*A.J.* §12). We obtain therefore an independent basis for the general theorem of §5.

7. By §§ 5–6, the orthogonal group $O^{(\mu)}_{m,p^n}$ contains a substitution T which replaces ξ_1 by $\lambda_1\xi_1 + \ldots\ldots + \lambda_m\xi_m$ for any set of solutions of

$$\lambda_1^2 + \lambda_2^2 + \ldots\ldots + \lambda_{m-1}^2 + \frac{1}{\mu}\lambda_m^2 = 1. \tag{10}$$

If S be any orthogonal substitution the product ST is orthogonal and therefore replaces ξ_1 by a function $\gamma_1\xi_1 + \ldots\ldots + \gamma_m\xi_m$ for which $\gamma_1^2 + \ldots\ldots + \gamma_{m-1}^2 + \frac{1}{\mu}\gamma_m^2 = 1$. Hence S replaces $\lambda_1\xi_1 + \ldots + \lambda_m\xi_m$ by $\gamma_1\xi_1 + \ldots\ldots + \gamma_m\xi_m$. It follows that the orthogonal group may be represented as a transitive substitution group on the N functions† $\lambda_1\xi_1 + \ldots\ldots + \lambda_m\xi_m$ in which $\lambda_1, \ldots\ldots, \lambda_m$ run through every set of solutions in the $GF[p^n]$ of equation (10). The isomorphism is in fact holoedric since among the above functions occur $\xi_1, \xi_2, \ldots\ldots, \xi_{m-1}$, as well as a function involving ξ_m. But an orthogonal substitution leaving these m functions all invariant is the identity.

Similarly, the subgroup of index 2 formed of the substitutions of $O^{(\mu)}_{m,p^n}$ having determinant unity may be represented as a transitive substitution group on the above N functions.

The orthogonal group contains the substitution C changing the signs of all m variables. The group of order 2 formed by C and the identity is evidently

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* Compare the intricate proof (for $n = 1$, $\mu = 1$) by Jordan, "Traité des substitutions," pp. 161–166, corrected and generalized to any n by the writer, Bull. Amer. Math. Soc., vol. 4, pp. 196–200; and to any μ, American Journal, vol. XXI, pp. 199–204. However, in the latter papers, the generators are found by the same investigation.

† The value of N is given in §9.

self-conjugate under $O^{(\mu)}_{m, p^n}$. The quotient-group will be designated $Q^{(\mu)}_{m, p^n}$. Combining into a single symbol $\{\lambda_1\xi_1 + \ldots + \lambda_m\xi_m\}$ the pair of linear functions *

$$\lambda_1\xi_1 + \ldots + \lambda_m\xi_m, \quad -\lambda_1\xi_1 - \ldots - \lambda_m\xi_m,$$

we obtain a set of $\frac{1}{2}N$ letters upon which $Q^{(\mu)}_{m, p^n}$ may be represented as a transitive substitution group, if the case $m = 3$, $p^n = 3$ be excluded. The isomorphism of the two groups is then seen to be holoedric [§16, case (1)].

For m even, the subgroup of orthogonal substitutions of determinant unity contains the substitution C. We obtain a quotient-group which may be represented as a transitive substitution group on the above $\frac{1}{2}N$ symbols.

For m odd, the subgroup of orthogonal substitutions of determinant unity does not contain C and may therefore be represented as a transitive substitution group on the $\frac{1}{2}N$ symbols, if $p^n > 3$ when $m = 3$.

8. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a set of marks of the $GF[p^n]$ such that

$$\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{m-1}^2 + \frac{1}{\mu}\lambda_m^2 = c, \tag{11}$$

where $c \neq 1$, there does not exist an orthogonal substitution replacing ξ_1 by $\omega \equiv \lambda_1\xi_1 + \lambda_2\xi_2 + \ldots + \lambda_m\xi_m$, but there may exist orthogonal substitutions which replace some other function

$$\omega' \equiv \lambda_1'\xi_1 + \lambda_2'\xi_2 + \ldots + \lambda_m'\xi_m,$$

by ω. From what follows, $\lambda_1', \ldots, \lambda_m'$ must also be a set of solutions of (11). By employing a new method of procedure, we may generalize the results of §7.

As in §3, the general orthogonal substitution A replaces the function $\lambda_1\xi_1 + \ldots + \lambda_m\xi_m$, where $\lambda_1, \ldots, \lambda_m$ satisfy (11), by

$$\sum_{j=1}^{m} \gamma_j\xi_j, \quad \gamma_j \equiv \sum_{i=1}^{m} \alpha_{ij}\lambda_i.$$

We prove that $\gamma_1, \ldots, \gamma_m$ satisfy relation (11). Indeed,

$$\sum_{j=1}^{m-1} \gamma_j^2 + \frac{1}{\mu}\gamma_m^2 = \sum_{i=1}^{m} \lambda_i^2 \left\{ \sum_{j=1}^{m-1} \alpha_{ij}^2 + \frac{1}{\mu}\alpha_{im}^2 \right\} + 2\sum_{i<k}^{1,\ldots,m} \lambda_i\lambda_k \left\{ \sum_{j=1}^{m-1} \alpha_{ij}\alpha_{kj} + \frac{1}{\mu}\alpha_{im}\alpha_{km} \right\}$$
$$= \sum_{i=1}^{m-1} \lambda_i^2 + \frac{1}{\mu}\lambda_m^2 = c,$$

* The proportionality factor of §1 must here be ± 1 in view of (10).

upon applying (1′) and (2′) and afterwards relation (11). *Hence the totality of functions* $\lambda_1\xi_1 + \lambda_2\xi_2 + \ldots + \lambda_m\xi_m$, *in which* $\lambda_1, \ldots, \lambda_m$ *are solutions of* (11), *are merely permuted by an arbitrary orthogonal substitution.*

9. In evaluating N, we consider more generally the number $N^{(\mu, c)}_{m, p^n}$ of sets of solutions $\lambda_1, \lambda_2, \ldots, \lambda_m$ in the $GF[p^n]$ of equation (11). This number is different according as m is even or odd, $\mu = 1$ or ν, $c = 0$, square or not-square. The value of the number is (*A. J.*, §3):

$$\begin{array}{lll} \text{For } m = 2M, & c \neq 0, & p^{n(2M-1)} \mp \varepsilon^M p^{n(M-1)}, \\ & c = 0, & p^{n(2M-1)} \pm \varepsilon^M (p^{nM} - p^{n(M-1)}). \\ \text{For } m = 2M+1, & c = 0, & p^{2nM}, \\ & c = \text{square}, & p^{2nM} \pm \varepsilon^M p^{nM}, \\ & c = \text{not-square} & p^{2nM} \mp \varepsilon^M p^{nM}, \end{array}$$

Here the upper or lower signs hold according as $\mu = 1$ or ν respectively; while ε denotes ± 1 according as $p^n = 4l \pm 1$, viz.,

$$\varepsilon \equiv (-1)^{\frac{p^n-1}{2}}$$

10. If $c \neq 0$, the sets of solutions of (11) have by pairs the same values of the ratios $\lambda_1 : \lambda_2 : \ldots : \lambda_m$, viz., one set of solutions and their negatives. As in §7, we obtain a representation of $O^{(\mu)}_{m, p^n}$ upon $N^{(\mu, c)}_{m, p^n}$ letters and of $Q^{(\mu)}_{m, p^n}$ upon $\frac{1}{2} N^{(\mu, c)}_{m, p^n}$ letters (using the results of §16).

If $c = 0$, we may reduce the number of letters exactly as in §1. We discard the function given by $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$, which is invariant under every orthogonal substitution. The remaining functions may be united into groups of $p^n - 1$ each, those of one group having the same ratios $\lambda_1 : \lambda_2 : \ldots : \lambda_m$. This is possible in view of the homogeneity of the orthogonal substitutions and of the relation (11) for $c = 0$. We thus obtain $(N^{(\mu, c)}_{m, p^n} - 1)/(p^n - 1)$ symbols $\{\lambda_1\xi_1 + \lambda_2\xi_2 + \ldots + \lambda_m\xi_m\}$, symbols which are merely permuted upon applying an arbitrary orthogonal substitution. Since

$$N^{(\mu, 0)}_{2M, p^n} - 1 \equiv (p^{nM} \mp \varepsilon^M)(p^{n(M-1)} \pm \varepsilon^M),$$

we may state the final result, incorporating the theorems of §§12–16:

The orthogonal quotient-group $Q^{(\mu)}_{m, p^n}$, $m > 2$, *may be represented as a transitive substitution group G on the following number of letters:*

$$\Omega \equiv (p^{n(m-1)} - 1)/(p^n - 1), \qquad (m \text{ odd}),$$

$$E \equiv [p^{nm/2} \mp (-1)^{m(p^n-1)/4}][p^{n(m/2-1)} \pm (-1)^{m(p^n-1)/4}] \div [p^n - 1] \quad (m \text{ even}),$$

the upper or lower signs holding according as $\mu = 1$ or $\mu = \nu$. For m odd, the group of orthogonal substitutions of determinant unity may be represented on Ω letters.

It remains to inquire which value of c gives a representation upon the lowest number of letters.

Suppose first that m is odd ($m \geqq 3$). For $c \neq 0$,* we may choose c among the squares or not-squares so that the number of letters is the least possible (§9), viz.:

$$\omega \equiv \tfrac{1}{2}\left[p^{n(m-1)} - p^{n(m-1)/2}\right].$$

For $c = 0$, the number of letters was seen to be

$$\Omega \equiv (p^{n(m-1)} - 1)/(p^n - 1).$$

If $p^n = 3$, then $\omega < \Omega$. If $p^n \geqq 5$, the condition $\omega > \Omega$ may be written

$$(p^n - 3)\,p^{n(m-1)} - p^{n(m+1)/2} + p^{n(m-1)/2} + 2 > 0,$$

and is always satisfied since $p^{n(m-1)} \geqq p^{n(m+1)/2}$, m being $\geqq 3$.

Suppose next that m is even. The case $m = 2$ is examined in §16, the orthogonal group being then commutative; as the case $c = 0$ leads to a trivial group G, a comparison with the case $c \neq 0$ is unnecessary. Suppose then that $m \geqq 4$. The various values of $c \neq 0$ give a representation on the same number of letters (§9):

$$\omega' \equiv \tfrac{1}{2}\left[p^{n(m-1)} \mp (-1)^{m(p^n-1)/4} p^{n(m/2-1)}\right].$$

For $c = 0$, the number of letters is given by the expression E of §10. In each case the upper or lower signs are to be taken according as $\mu = 1$ or $\mu = \nu$. Consider the lowest value ω_1' of ω' and the greatest value E_1 of E, viz.:

$$\omega_1' \equiv \tfrac{1}{2}\left[p^{n(m-1)} - p^{n(m/2-1)}\right], \quad E_1 \equiv \left[p^{nm/2} - 1\right]\left[p^{n(m/2-1)} + 1\right] \div \left[p^n - 1\right].$$

We proceed to prove that if $p^n > 3$, $\omega_1' > E_1$. The condition is seen to be

$$p^{n(m-1)}(p^n - 3) - 3p^{nm/2} + 3p^{n(m/2-1)} + 2 > 0.$$

Since $m \geqq 4$, $p^n \geqq 5$, we have

$$p^{n(m-1)}(p^n - 3) \geqq 2p^{n(m-1)} \geqq 10p^{n(m-2)} \geqq 10p^{nm/2}.$$

The condition is, therefore, satisfied. Hence, $\omega' > E$ if $p^n > 3$. For $p^n = 3$, the question is more delicate. Since

$$\omega' = \tfrac{1}{2}\,3^{m/2-1}\left[3^{m/2} \mp (-1)^{m/2}\right], \quad E = \tfrac{1}{2}\left[3^{m/2} \mp (-1)^{m/2}\right]\left[3^{m/2-1} \pm (-1)^{m/2}\right],$$

* For $p^n = 3$, $m = 3$, $c \neq 0$, the isomorphism is not holoedric (§16), so that the representation fails.

we have $\omega' > E$ or $\omega' < E$ according as $\pm(-1)^{m/2}$ is -1 or $+1$. Hence, for $p^n = 3$, $\omega' > E$ in the two cases $\mu = 1$ with $m/2$ odd, $\mu = \nu$ with $m/2$ even; while $\omega' < E$ when $\mu = 1$ with $m/2$ even, or $\mu = \nu$ with $m/2$ odd.

We combine our results into the theorem:

If $m > 2$, the number of letters employed in the above representation of the quotient-group $Q^{(\mu)}_{m, p^n}$ as a transitive substitution group is less for the case $c = 0$ than for any $c \neq 0$, except for the cases $p^n = 3$, m odd, $m > 3$; $p^n = 3$, $m/2$ even, $\mu = 1$; $p^n = 3$, $m/2$ odd, $\mu = \nu$. For the latter cases the number of letters, given by a suitably chosen mark $c \neq 0$, is*

$$\tfrac{1}{2}\left[3^{m-1} - 3^{(m-1)/2}\right] \textit{ for } m \textit{ odd}, m > 3; \quad \tfrac{1}{2}\left[3^{m-1} - 3^{m/2-1}\right] \textit{ for } m \textit{ even}.$$

12. Theorem.—*The orthogonal group $O^{(\mu)}_{m, p^n}$ permutes transitively the totality of symbols $\{\lambda_1\xi_1 + \lambda_2\xi_2 + \ldots + \lambda_m\xi_m\}$ in which $\lambda_1, \ldots, \lambda_m$ are solutions of equation (11), c being any fixed mark of the $GF[p^n]$.*

We are to prove that there exists an orthogonal substitution A replacing the function $\omega' \equiv \lambda_1'\xi_1 + \lambda_2'\xi_2 + \ldots + \lambda_m'\xi_m$, where $\lambda_1', \ldots, \lambda_m'$ form a particular set of solutions in the $GF[p^n]$, $p \neq 2$, of

$$\lambda_1'^2 + \lambda_2'^2 + \ldots + \lambda_{m-1}'^2 + \frac{1}{\mu}\lambda_m'^2 = c, \tag{12}$$

by the function $\omega \equiv \lambda_1\xi_1 + \lambda_2\xi_2 + \ldots + \lambda_m\xi_m$, where $\lambda_1, \ldots, \lambda_m$ form an arbitrary set of solutions not all zero of the analogous relation (11).

If c is the square of a mark $\gamma \neq 0$ of the field, we may take $\lambda_1' = \gamma$, $\lambda_2' = 0, \ldots, \lambda_m' = 0$. By §§5–6, there exists an orthogonal substitution A replacing ξ_1 by

$$\frac{\lambda_1}{\gamma}\xi_1 + \frac{\lambda_2}{\gamma}\xi_2 + \ldots + \frac{\lambda_m}{\gamma}\xi_m,$$

$$\left(\frac{\lambda_1}{\gamma}\right)^2 + \left(\frac{\lambda_2}{\gamma}\right)^2 + \ldots + \left(\frac{\lambda_{m-1}}{\gamma}\right)^2 + \frac{1}{\mu}\left(\frac{\lambda_m}{\gamma}\right)^2 = 1,$$

in view of relation (11) and $\gamma^2 = c$. Hence, A replaces $\omega' \equiv \lambda\xi_1$ by ω.

Suppose next that c is zero or a not-square. By §9, there exist in the $GF[p^n]$ more than one set of solutions of†

$$D^2 + E^2 = c \tag{13}$$

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* For $c = 0$, the number of letters is given by the theorem of §10.

† For $m = 2$, E^2 is to be replaced by $\frac{1}{\mu}E^2$ in (13) and 1 by $\frac{1}{\mu}$ in the right member of the second equation (15). But the final equation is exactly (16).

except in the case $c=0$, -1 a not-square, when the only solution is evidently $D=E=0$. Excluding this case for the present, we proceed to determine an orthogonal substitution A which replaces $D\xi_1+E\xi_2$, where D and E satisfy (13) and $E\neq 0$, by the function ω. The conditions are evidently

$$D\alpha_{1j}+E\alpha_{2j}=\lambda_j, \qquad (j=1,2,\ldots,m \quad (14)$$

together with the conditions that A shall be orthogonal.

If α_{1j}, α_{2j} be determined so that (14) and

$$\sum_{j=1}^{m-1}\alpha_{1j}^2+\frac{1}{\mu}\alpha_{1m}^2=1,\quad \sum_{j=1}^{m-1}\alpha_{2j}^2+\frac{1}{\mu}\alpha_{2m}^2=1,\quad \sum_{j=1}^{m-1}\alpha_{1j}\alpha_{2j}+\frac{1}{\mu}\alpha_{1m}\alpha_{2m}=0 \qquad (15)$$

are satisfied, it follows from §5 that marks

$$\alpha_{ij} \qquad (i=3,\ldots,m;\ j=1,2,\ldots,m)$$

may be found such that A will be the required orthogonal substitution. Substituting the values of α_{2j} determined by (14) in the second and third relations (15), the latter become

$$D^2\Big(\sum_{j=1}^{m-1}\alpha_{1j}^2+\frac{1}{\mu}\alpha_{1m}^2\Big)-2D\Big(\sum_{j=1}^{m-1}\alpha_{1j}\lambda_j+\frac{1}{\mu}\alpha_{1m}\lambda_m\Big)+\sum_{j=1}^{m-1}\lambda_j^2+\frac{1}{\mu}\lambda_m^2=E^2,$$

$$D\Big(\sum_{j=1}^{m-1}\alpha_{1j}^2+\frac{1}{\mu}\alpha_{1m}^2\Big)-\quad\Big(\sum_{j=1}^{m-1}\alpha_{1j}\lambda_j+\frac{1}{\mu}\alpha_{1m}\lambda_m\Big)=0.$$

In view of (11), (13) and the first relation (15), these two conditions reduce to the single condition

$$\sum_{j=1}^{m-1}\alpha_{1j}\lambda_j+\frac{1}{\mu}\alpha_{1m}\lambda_m=D. \qquad (16)$$

The conditions are therefore (16) and the first relation (15), say (15′). These two conditions can always be satisfied.

If $\lambda_1=\lambda_2=\ldots=\lambda_{m-1}=0$, then $\lambda_m\neq 0$ by hypothesis, so that (11) gives $\frac{1}{\mu}\lambda_m^2=c\neq 0$. Then (16) becomes $\frac{1}{\mu}\alpha_{1m}\lambda_m=D$, which determines α_{1m}. Condition (15) is the only condition upon $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m-1}$; it may be solved in the field by §9.

In the contrary case, we may take $\lambda_1\neq 0$ in view of the symmetry of (11) and (16) in $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$. Eliminating α_{11} between (16) and (15′), the latter may be replaced by

$$\lambda_1^2\Big(\sum_{j=2}^{m-1}\alpha_{1j}^2+\frac{1}{\mu}\alpha_{1m}^2\Big)+\Big(\sum_{j=2}^{m-1}\alpha_{1j}\lambda_j+\frac{1}{\mu}\alpha_{1m}\lambda_m-D\Big)^2=\lambda_1^2. \qquad (17)$$

If $D=0$, then $c=E^2 \neq 0$, so that the problem, is solved as above.

For the case $m=2$, (11) and (13) have the form

$$D^2+\frac{1}{\mu}E^2=c, \quad \lambda_1^2+\frac{1}{\mu}\lambda_2^2=c.$$

The condition (17) may be written in the successive forms

$$\frac{1}{\mu}\left(\lambda_1^2+\frac{1}{\mu}\lambda_2^2\right)\alpha_{12}^2-\frac{2}{\mu}D\lambda_2\alpha_{12}=\lambda_1^2-D^2,$$
$$(c\alpha_{12}-D\lambda_2)^2=\mu c\,(\lambda_1^2-D^2)+D^2\lambda_2^2=\lambda_1^2E^2.$$

If $c=0$, the first form of the relation determines α_{12} linearly unless $D\lambda_2=0$, a case evidently excluded. If $c \neq 0$, the second relation determines α_{12} in the field. Then (16) determines α_{11} in the field.

13. For $m=3$, relation (17) may be written

$$\alpha_{12}^2\left(\lambda_1^2+\lambda_2^2\right)+\frac{1}{\mu}\alpha_{1m}^2\left(\lambda_1^2+\frac{1}{\mu}\lambda_m^2\right)+\frac{2}{\mu}\alpha_{12}\,\alpha_{1m}\,\lambda_2\,\lambda_m-2D\,\alpha_{12}\,\lambda_2$$
$$-\frac{2}{\mu}D\alpha_{1m}\,\lambda_m=\lambda_1^2-D^2. \qquad (17_3)$$

If $\lambda_1^2+\lambda_2^2=0$, then $\lambda_2 \neq 0$ and the equation determines α_{12} linearly, its coefficient not vanishing for every α_{1m}. In the contrary case, we multiply (17_3) by $\lambda_1^2+\lambda_2^2$ and obtain the equivalent equation

$$\left\{\alpha_{12}\left(\lambda_1^2+\lambda_2^2\right)+\frac{1}{\mu}\alpha_{1m}\,\lambda_2\,\lambda_m-D\lambda_2\right\}^2$$
$$=-\frac{1}{\mu}\alpha_{1m}^2\,\lambda_1^2\,c+\frac{2}{\mu}D\alpha_{1m}\,\lambda_m\,\lambda_1^2+\lambda_1^2\left(\lambda_1^2+\lambda_2^2-D^2\right).$$

If $c=0$, then $\lambda_m \neq 0$, so that α_{1m} may be chosen to make the right member a square in the field. If $c \neq 0$, the right member may be written

$$-\frac{1}{\mu}c\lambda_1^2\left(\alpha_{1m}-Dc^{-1}\lambda_m\right)^2+\lambda_1^2\left(\lambda_1^2+\lambda_2^2-D^2+\frac{1}{\mu}D^2\lambda_m^2c^{-1}\right).$$

Applying (11) and (13) to reduce the last term, the condition becomes

$$\left\{\alpha_{12}\left(\lambda_1^2+\lambda_2^2\right)+\frac{1}{\mu}\alpha_{1m}\,\lambda_2\,\lambda_m-D\lambda_2\right\}^2$$
$$+\frac{1}{\mu}c\lambda_1^2\left\{\alpha_{1m}-Dc^{-1}\lambda_m\right\}^2=c^{-1}E^2\lambda_1^2\left(\lambda_1^2+\lambda_2^2\right).$$

This equation has solutions in the field for the two quantities in brackets (*A. J.*,

§3) and therefore (17_3) has solutions α_{12}, α_{1m}. Then (16) determines α_{11} in the field.

14. The method of procedure for $m > 3$ will be illustrated by the case $m = 4$. If $\lambda_3 = 0$, the problem reduces to the case just solved. Suppose therefore that $\lambda_3 \neq 0$. The three sums

$$\lambda_2^2 + \lambda_3^2, \quad \lambda_2^2 + \frac{1}{\mu}\lambda_m^2, \quad \lambda_3^2 + \frac{1}{\mu}\lambda_m^2$$

do not all vanish (p being $\neq 2$). If $\lambda_2^2 + \lambda_3^2 \neq 0$, we determine α_{13} from

$$\alpha_{12}\lambda_2 + \alpha_{13}\lambda_3 = 0,$$

whence (17) takes the form

$$\lambda_1^2\alpha_{12}^2\left(1 + \lambda_2^2/\lambda_3^2\right) + \frac{1}{\mu}\alpha_{1m}^2\left(\lambda_1^2 + \frac{1}{\mu}\lambda_m^2\right) - \frac{2}{\mu}D\alpha_{1m}\lambda_m = \lambda_1^2 - D^2.$$

If the coefficient of α_{1m}^2 vanishes, then $\lambda_m \neq 0$ and the equation determines α_{1m} linearly and therefore in the field. In the contrary case we complete the square in α_{1m}, when the left member becomes the sum of two squares each multiplied by marks $\neq 0$. Solutions* α_{12}, α_{1m} therefore exist in the field (*A. J.*, §3).

If $\lambda_2^2 + \frac{1}{\mu}\lambda_m^2 \neq 0$, we determine α_{1m} so that

$$\alpha_{12}\lambda_2 + \frac{1}{\mu}\alpha_{1m}\lambda_m = 0,$$

whence (17) becomes

$$\lambda_1^2\alpha_{12}^2(1 + \mu\lambda_2^2/\lambda_m^2) + \alpha_{13}^2(\lambda_1^2 + \lambda_3^2) - 2D\lambda_3\alpha_{13} = \lambda_1^2 - D^2.$$

Proceeding as before, we obtain solutions α_{12}, α_{13} in the field.

To give a treatment which shall include the case $c = 0, -1$ a not-square (not solved by the preceding method), we start with a linear function

$$\omega' \equiv D\xi_1 + E\xi_2 + F\xi_3,$$

where
$$D^2 + E^2 + \tau F^2 = c, \qquad (18)$$
$$\left(\tau = 1 \text{ for } m > 3, \quad \tau = \frac{1}{\mu} \text{ for } m = 3\right)$$

solutions of which exist (*A. J.*, §3), such that $F \neq 0$. In order that a linear substitution A shall replace ω' by $\lambda_1\xi_1 + \ldots + \lambda_m\xi_m$, in which the λ_i are solutions of (11), the following conditions must be satisfied:

$$D\alpha_{1j} + E\alpha_{2j} + F\alpha_{3j} = \lambda_j, \quad (j = 1, 2, \ldots, m). \quad (19)$$

* If $\lambda_2^2 + \lambda_3^2 = 0$, the condition becomes $(c\alpha_{1m} - D\lambda_m)^2 = \mu E^2\lambda_1^2$ and has solutions only for $\mu =$ square. For this reason we treat the three cases in succession.

We impose also the condition that A shall be orthogonal. In view of §5, we need only consider the following orthogonal conditions:*

$$\sum_{j=1}^{m-1} \alpha_{ij}^2 + \frac{1}{\mu}\,\alpha_{im}^2 = 1 \quad (i = 1,\, 2), \quad \sum_{j=1}^{m-1} \alpha_{3j}^2 + \frac{1}{\mu}\,\alpha_{3m}^2 = \begin{cases} 1 & (m > 3), \\ 1/\mu & (m = 3), \end{cases} \tag{20}$$

$$\sum_{j=1}^{m-1} \alpha_{1j}\alpha_{2j} + \frac{1}{\mu}\,\alpha_{1m}\alpha_{2m} = 0, \quad \sum \alpha_{1j}\alpha_{3j} + \frac{1}{\mu}\,\alpha_{1m}\alpha_{3m} = 0,$$

$$\sum \alpha_{2j}\alpha_{3j} + \frac{1}{\mu}\,\alpha_{2m}\alpha_{3m} = 0. \tag{21}$$

Substituting the value of α_{3j} determined by (19) in (20) and (21), we obtain the the reduced set of conditions:†

$$\sum_{j=1}^{m-1} \alpha_{1j}\lambda_j + \frac{1}{\mu}\,\alpha_{1m}\lambda_m = D, \quad \sum_{j=1}^{m-1} \alpha_{2j}\lambda_j + \frac{1}{\mu}\,\alpha_{2m}\lambda_m = E. \tag{22}$$

$$\sum_{j=1}^{m-1} \alpha_{1j}^2 + \frac{1}{\mu}\,\alpha_{1m}^2 = 1, \sum_{j=1}^{m-1} \alpha_{2j}^2 + \frac{1}{\mu}\,\alpha_{2m}^2 = 1, \quad \sum_{j=1}^{m-1} \alpha_{1j}\alpha_{2j} + \frac{1}{\mu}\,\alpha_{1m}\alpha_{2m} = 0. \tag{23}$$

For example, the second condition (21) becomes

$$\sum_{j=1}^{m-1} \alpha_{1j}\lambda_j + \frac{1}{\mu}\,\alpha_{1m}\lambda_m - D\Big(\sum \alpha_{1j}^2 + \frac{1}{\mu}\,\alpha_{1m}^2\Big) - E\Big(\sum \alpha_{1j}\alpha_{2j} + \frac{1}{\mu}\,\alpha_{1m}\alpha_{2m}\Big) = 0.$$

The third condition (21) takes the form

$$\begin{aligned} \Big(\sum \lambda_j^2 + \frac{1}{\mu}\,\lambda_m^2\Big) - 2D\Big(\sum \alpha_{1j}\lambda_j + \frac{1}{\mu}\,\alpha_{1m}\lambda_m\Big) - 2E\Big(\sum \alpha_{2j}\lambda_j + \alpha_{2m}\lambda_m\Big) \\ + D^2\Big(\sum \alpha_{1j}^2 + \frac{1}{\mu}\,\alpha_{1m}^2\Big) + E^2\Big(\sum \alpha_{2j}^2 + \frac{1}{\mu}\,\alpha_{2m}^2\Big) \\ + 2DE\Big(\sum \alpha_{1j}\alpha_{2j} + \frac{1}{\mu}\,\alpha_{1m}\alpha_{2m}\Big) = \tau F^2, \end{aligned}$$

where $\tau = 1$ or $1/\mu$ according as $m > 3$ or $m = 3$, the summations all extending from $j = 1, 2, \ldots, m-1$. Upon substituting for the quantities in parentheses their values, the equation reduces to (18). If any λ_i is zero, say $\lambda_1 = 0$, we take $\alpha_{11} = \alpha_{21} = 0$ and have a similar set of equations with m replaced by $m - 1$.

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* For $m = 3$, $\mu = \nu$, we must replace 1 by $1/\mu$ in the right member of (20) for $i = 3$, and similarly F^2 by $\frac{1}{\mu}F^2$ in (18).

† These conditions may be also obtained by requiring the inverse A^{-1} to replace $\lambda_1\xi_1 + \ldots + \lambda_m\xi_m$ by $D\xi_1 + E\xi_2 + F\xi_3$, the remaining $m - 2$ conditions being then satisfied in virtue of (19).

In proving the existence in the $GF[p^n]$ of a set of solutions α_{1j}, α_{2j} of equations (22) and (23), we limit the discussion, for the sake of simplicity, to the case $m=3$. Taking $\lambda_1 \neq 0$, we eliminate α_{11} and α_{21} from the three equations (23) by means of (22). We obtain equation (17_3) of §13 and a similar one with α_{12}, α_{1m}, D replaced by α_{22}, α_{2m} E respectively, and the third equation

$$\alpha_{12}\alpha_{22}(\lambda_1^2+\lambda_2^2)+\frac{1}{\mu}\alpha_{1m}\alpha_{2m}\left(\lambda_1^2+\frac{1}{\mu}\lambda_m^2\right)+\frac{1}{\mu}\lambda_2\lambda_m(\alpha_{12}\alpha_{2m}+\alpha_{22}\alpha_{1m})$$
$$-\lambda_2(D\alpha_{22}+E\alpha_{12})-\frac{1}{\mu}\lambda_m(D\alpha_{2m}+E\alpha_{1m})+DE=0. \quad (24)$$

If $\lambda_1^2+\lambda_2^2=0$, $\lambda_1^2+\frac{1}{\mu}\lambda_m^2=0$, so that $-\frac{1}{\mu}$ is a square, we have, by addition and application of (11), $\lambda_1^2+c=0$. Hence $c\neq 0$. The three conditions may be written

$$xy=\tfrac{1}{2}(D^2-c),\quad zw=\tfrac{1}{2}(E^2-c),\quad xw+yz=DE, \quad (25)$$

where we have set

$$x\equiv\alpha_{12}\lambda_2-D,\quad y\equiv\frac{1}{\mu}\alpha_{1m}\lambda_m-D,\quad z\equiv\alpha_{22}\lambda_2-E,\quad w\equiv\frac{1}{\mu}\alpha_{2m}\lambda_m-E.$$

Eliminating y and z from (25), we find

$$\{x(E^2-c)-DEz\}^2+z^2\frac{c}{\mu}F^2=0. \quad (26)$$

Since $D^2-c=0$ or $E^2-c=0$ requires that c be a square, when the problem has been solved in §12, equation (26) leads to solutions z and x different from zero in the field, from which y and w are determined by (25).

If the above sums do not both vanish, their analogous rôle in the formulæ enable us to assume that $\lambda\equiv\lambda_1^2+\lambda_2^2\neq 0$. As in §13, the first and second conditions may be written in the form

$$X^2+\frac{1}{\mu}c\lambda_1^2Y^2=\lambda_1^2E\lambda c^{-1},\quad Z^2+\frac{1}{\mu}c\lambda_1^2W^2=\lambda_1^2D\lambda c^{-1}, \quad (27)$$

where have set

$$X\equiv\lambda\alpha_{12}+\frac{1}{\mu}\alpha_{1m}\lambda_2\lambda_m-D\lambda_2,\quad Y\equiv\alpha_{1m}-\lambda_mDc^{-1},$$
$$Z\equiv\lambda\alpha_{22}+\frac{1}{\mu}\alpha_{2m}\lambda_2\lambda_m-E\lambda_2,\quad W\equiv\alpha_{2m}-\lambda_mEc^{-1}.$$

Multiplying the third condition (24) by $\lambda\equiv\lambda_1^2+\lambda_2^2$, it may be written

$$XZ+\frac{1}{\mu}c\lambda_1^2YW=\lambda_1^2DE(\lambda_m^2c^{-1}\mu^{-1}-1)\equiv\kappa. \quad (28)$$

For brevity, put $\rho \equiv \frac{1}{\mu} c\lambda_1^2$, $\sigma = \lambda_1^2\lambda c^{-1}$. Then (27) and (28) give

$$\rho^2 Y^2 W^2 = (E\sigma - X^2)(D\sigma - Z^2) = (\varkappa - XZ)^2.$$

From the latter equality,

$$(D\sigma X - \varkappa Z)^2 + Z^2(DE\sigma^2 - \varkappa^2) = D\sigma\,(DE\sigma^2 - \varkappa^2). \tag{29}$$

Applying (11), we find

$$DE\sigma^2 - \varkappa^2 = \lambda^2\lambda_1^4 c^{-2} DE\,(1 - DE).$$

In view of $\rho W^2 = D\sigma - Z^2$; equation (29) gives

$$W^2\,(DE\sigma^2 - \varkappa^2)\,\rho = (D\sigma X - \varkappa Z)^2. \tag{30}$$

The coefficient of W^2 may, for $DE \neq 0$, be supposed to be a square, by proper choice of the particular set of solutions of (18). For each set of solutions Z, X of (29), W is determined by (30) and then Y is determined linearly by (28). The required coefficients $\alpha_{1m}, \alpha_{2m}, \alpha_{12}, \alpha_{22}$ are then determined in the field.

15. THEOREM.—*The orthogonal group $O^{(\mu)}_{m,\,p^n}$, $m > 2$, is holoedrically isomorphic with the transitive substitution group on the totality of linear functions $\lambda_1\xi_1 + \lambda_2\xi_2 + \ldots + \lambda_m\xi_m$ in which*

$$\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{m-1}^2 + \frac{1}{\mu}\lambda_m^2 = c = \textit{constant}. \tag{31}$$

We prove that the identity is the only orthogonal substitution which leaves fixed all of these linear functions. In order that A leave fixed $\lambda_1\xi_1 + \ldots + \lambda_m\xi_m$, it is necessary that

$$\lambda_1\alpha_{1j} + \lambda_2\alpha_{2j} + \ldots + \lambda_m\alpha_{mj} = \lambda_j, \quad (j = 1, \ldots, m). \tag{32}$$

If there exist m linearly independent sets of solutions of (31),

$$\lambda_{1j},\ \lambda_{2j},\ \ldots,\ \lambda_{mj} \qquad (j = 1, \ldots, m)$$

such, therefore, that their determinant $(\lambda_{ij}) \neq 0$, then (32) gives

$$\alpha_{1j} = \alpha_{2j} = \ldots = \alpha_{jj} - 1 = \ldots = \alpha_{mj} = 0, \quad (j = 1, 2, \ldots, m),$$

so that A would reduce to the identity.

For the case c, a square, say σ^2, in the $GF[p^n]$, we may take the sets

$$\sigma, 0, 0, \ldots, 0;\ 0, \sigma, 0, \ldots, 0;\ \ldots;\ 0, 0, \ldots, \sigma, 0;\ \lambda_1, \lambda_2, \ldots, \lambda_m,$$

where $\lambda_1, \ldots, \lambda_m$ is any set of solutions in which $\lambda_m \neq 0$. The determinant of these sets is $\sigma^{m-1}\lambda_m \neq 0$.

For c a not-square and $\mu = \nu$, a not-square, we may take as one set $0, 0, \ldots, 0, x$, where $\frac{1}{\nu} x^2 = c$, or $x^2 = \nu c =$ square. Hence, the problem is reduced to the case of $m-1$ sets of solutions of $\lambda_1^2 + \ldots + \lambda_{m-1}^2 = c$, since the determinant

$$\begin{vmatrix} \lambda_{11} & \lambda_{21} & \ldots & \lambda_{m-1\,1} & 0 \\ \lambda_{12} & \lambda_{22} & \ldots & \lambda_{m-1\,2} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_{1\,m-1} & \lambda_{2\,m-1} & \ldots & \lambda_{m-1\,m-1} & 0 \\ 0 & 0 & \ldots & 0 & x \end{vmatrix} \equiv |\lambda_{ij}| \cdot x$$

will not vanish if $m-1$ independent sets $\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{m-1j}$ exist. Hence, for $c =$ not-square, the problem reduces to the case $\mu = 1$, $m \geqq 2$,

For $m = 2$, we take the sets λ_1, λ_2 and $-\lambda_2, \lambda_1$ of determinant

$$\lambda_1^2 + \lambda_2^2 = c \neq 0.$$

For $m = 3$,* we take the three sets given by the rows of

$$\begin{vmatrix} \lambda_1 & \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 & 0 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{vmatrix} \equiv \lambda_3'(\lambda_1^2 + \lambda_2^2)$$

where $\lambda_1^2 + \lambda_2^2 = c$, $\lambda_1'^2 + \lambda_2'^2 + \lambda_3'^2 = c$, $\lambda_3' \neq 0$. Such solutions always exist (*A. J.*, §3). The method of procedure from the case $m = r$ to $m = r + 1$ is apparent.

There remains the case $c = 0$. Let first $m = 3$. If -1 be a square in the field, $\lambda_1^2 + \lambda_2^2 = 0$ has solutions $\lambda_1 \neq 0$. We then take

$$\begin{vmatrix} \lambda_1 & \lambda_2 & 0 \\ \lambda_2 & \lambda_1 & 0 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{vmatrix} = \lambda_3'(\lambda_1^2 - \lambda_2^2) = 2\lambda_3'\lambda_1^2 \neq 0.$$

If -1 be a not-square and $\mu = \nu$, a not-square, there exist solutions in the field of $\lambda_2^2 + \frac{1}{\nu}\lambda_3^2 = 0$. Then $(\lambda_3/\nu)^2 + \frac{1}{\nu}\lambda_2^2 = 0$. We thus take

$$\begin{vmatrix} \lambda_1' & \lambda_2' & \lambda_3' \\ 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_3/\nu & \lambda_2 \end{vmatrix} \equiv 2\lambda_1'\lambda_2^2 \neq 0.$$

* It may be shown that it is possible to choose solutions of $\lambda_1^2 + \lambda_2^2 = c$, so that the three sets $\lambda_1, \lambda_2, 0$; $0, \lambda_1, \lambda_2$; $\lambda_2, 0, \lambda_1$ are linearly independent, i. e., such that $\lambda_1^3 + \lambda_2^3 \neq 0$.

The method employed in these two cases may be extended by induction to the case of general m.

Let next $c=0$, $\mu=1$, $-1=$ not square in the $GF[p^n]$.

For $m=3$, $\lambda_1^2+\lambda_2^2+\lambda_3^2=0$ has p^{2n} sets of solutions in the $GF[p^n]$. We prove that it is possible to choose a set $\lambda_1, \lambda_2, \lambda_3$ such that

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_3 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 & \lambda_1 \end{vmatrix} \equiv (\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2-\lambda_1\lambda_2-\lambda_1\lambda_3-\lambda_2\lambda_3),$$

shall not vanish. Since $\lambda_1^2+\lambda_2^2+\lambda_3^2=0$, the second factor vanishes if and only if the first factor $\lambda_1+\lambda_2+\lambda_3$ vanishes. But if the latter vanishes then

$$\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3=0$$

so that either $\lambda_2=\lambda_3=0$ (which case may be excluded) or else λ_3/λ_2 is a root of $\omega^2+\omega+1=0$ in the field, where, by the symmetry of the equations, we have taken $\lambda_2 \neq 0$. In the second case,

$$\lambda_3=\omega\lambda_2, \quad \lambda_1=-\lambda_2-\lambda_3=\omega^2\lambda_2.$$

Hence, of the $p^{2n}-1$ sets of solution not all zero, at most $2(p^n-1)$ are to be excluded.

For $m=4$, we obtain from the preceding sets and any fourth set λ_i' in which $\lambda_1'\neq 0$ the required independent sets:

$$\begin{vmatrix} \lambda_1' & \lambda_2' & \lambda_3' & \lambda_4' \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_3 & \lambda_1 & \lambda_2 \\ 0 & \lambda_2 & \lambda_3 & \lambda_1 \end{vmatrix} \neq 0.$$

By an evident induction, m independent sets of m solutions exist.

16. It remains to study the isomorphism of the orthogonal quotient-group $Q_{m,p^n}^{(\mu)}$ with the substitution group G on the symbols $\{\lambda_1\xi_1+\dots+\lambda_m\xi_m\}$ in which $\lambda_1, \dots, \lambda_m$ satisfy (31).

Let first $m=2$. The case $c=0$, $-1/\mu=$ not-square must be excluded since the only solutions are then $\lambda_1=\lambda_m=0$, whence G is the identity. If $-\dfrac{1}{\mu}=$ square, λ_m may be chosen arbitrarily $\neq 0$, when λ_1 has two values; the $2(p^n-1)$ sets of solutions yield two symbols, so that G is of order 2. These results agree with those of §10 for $m=2$, since E is then 0 or 2 according as $\pm\epsilon=-1$ or $+1$. For $m=2$, $c\neq 0$, the number of symbols is $\frac{1}{2}(p^n\mp\epsilon)$ according as

$\mu = 1$ or ν, where $\varepsilon = (-1)^{(p^n-1)/2}$. The order of the binary orthogonal group of determinant unity is $p^n \mp \varepsilon$ and the group is commutative (*A. J.*, p. 207). Considering the self-conjugate subgroup formed by the identity and the substitution changing the signs of ξ_1 and ξ_m, the quotient-group $Q_{2,p^n}^{(\mu)}$ is a commutative group of order and degree $\frac{1}{2}(p^n \mp \varepsilon)$. By §12 it is transitive. The substitution group is therefore *regular.** This result follows also directly from the remark at the end of §3 in connection with §6, since for $m = 2$ it is indifferent whether we employ the symbol $[\lambda_1, \lambda_2]$ or the matrix symbol.

Let next $m \overline{\gt} 3$. The holoedric isomorphism of the two groups will follow when the only substitutions leaving every symbol fixed are the identity and the substitution C, which changes the sign of every index ξ_i. Suppose therefore that A is an orthogonal substitution which leaves every symbol fixed.

Case (1): $c =$ square $= \sigma^2$. Among the symbols occurs $\{\sigma\xi_1\}$. Then A_1 where A_1 is either A itself or else the product AC, will replace $\sigma\xi_1$ by $+\sigma\xi_1$, so that A_1 leaves ξ_1 fixed.

Among the $p^n - \varepsilon$ sets of solutions in the $GF[p^n]$ of

$$\lambda_1^2 + \lambda_2^2 = \sigma^2, \quad \varepsilon \equiv (-1)^{(p^n-1)/2},$$

occur the sets $\lambda_1 = \pm\sigma$, $\lambda_2 = 0$; $\lambda_1 = 0$, $\lambda_2 = \pm\sigma$. Hence, for $p^n > 5$, there exist solutions λ_1, λ_2 both different from zero. The corresponding symbol $\{\lambda_1\xi_1 + \lambda_2\xi_2\}$ must be unaltered by A_1. But if A_1 multiply $\lambda_1\xi_1 + \lambda_2\xi_2$ by -1 then A_1 would replace $-\lambda_1\xi_1 + \lambda_2\xi_2$ by $-3\lambda_1\xi_1 + \lambda_2\xi_2$. Since the former function leads to a symbol, so must also the latter, whence $9\lambda_1^2 + \lambda_2^2 = c$, requiring $8\lambda_1^2 = 0$ and therefore $\lambda_1 = 0$. This being impossible, A_1 must leave $\lambda_1\xi_1 + \lambda_2\xi_2$ fixed and therefore also ξ_2. Similarly, A_1 leaves fixed $\xi_3, \ldots, \xi_{m-1}$, and, if $\mu = 1$, also ξ_m, so that A_1 is the identity. If $\mu = \nu$, a not-square, there exist $p^n + \varepsilon$ sets of solutions of

$$\lambda_1^2 + \frac{1}{\mu}\lambda_m^2 = \sigma^2.$$

Excluding the sets $\lambda_1 = \pm\sigma$, $\lambda_m = 0$, there remain, if $p^n > 3$, sets with $\lambda_1 \neq 0$, $\lambda_m \neq 0$. Hence, for $p^n > 3$, A_1 leaves ξ_m fixed.

For $p^n = 5$, $\mu = \nu$, it has been shown that A_1 leaves ξ_1 and ξ_m fixed. Since there are solutions of $\lambda_i^2 + \frac{1}{\mu}\lambda_m^2 = \sigma^2$ with $\lambda_i \neq 0$, $\lambda_m \neq 0$, it follows that A_1 leaves ξ_i fixed. Hence, A_1 is the identity.

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* Burnside, "The Theory of Groups," p. 177, Cor. III.

For $p^n = 5$, $\mu = 1$, we prove that A_1 leaves fixed

$$\sigma(\xi_1 + 2\xi_2 + \xi_3), \quad \sigma(\xi_1 - 2\xi_2 + \xi_3),$$

each leading to a symbol, so that A_1 leaves fixed their difference $4\sigma\xi_2$ and, therefore, ξ_2 and hence also ξ_3. In fact, if A_1 multiplies $\sigma(\xi_1 + 2\xi_2 + \xi_3)$ by -1, then A_1 replaces $\sigma(-\xi_1 + 2\xi_2 + \xi_3)$ by $\sigma(-3\xi_1 - 2\xi_2 - \xi_3)$. Since the former leads to a symbol so must the latter, whence $14\sigma^2 \equiv \sigma^2 \pmod 5$. This being impossible, A_1 leaves fixed ξ_1, ξ_2, ξ_3, and similarly $\xi_4, \ldots, \xi_m$.

For $p^n = 3$, $\mu = -1$, a not-square, we consider the functions $\xi_1+\xi_2+\xi_m$ and $\xi_1 - \xi_2 + \xi_m$, each leading to a symbol, and find that A_1 must multiply each by $+1$ and hence leave ξ_2 and ξ_m fixed. Similarly, A_1 leaves $\xi_3, \ldots, \xi_{m-1}$ fixed and is the identity.

For $p^n = 3$, $\mu = 1$, the method fails, if $m = 3$, since two of the three solutions of $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$ must be zero. Then A_1 is not necessarily the identity but may be the orthogonal substitution changing the signs of ξ_2 and ξ_3 and still leave every symbol unaltered.

For $p^n = 3$, $\mu = 1$, $m > 3$, A_1 must leave fixed the functions

$$\xi_1, \quad \xi_1 + \xi_2 + \xi_3 + \xi_4, \quad \xi_1 + \xi_2 + \xi_3 - \xi_4, \quad \xi_1 + \xi_2 - \xi_3 + \xi_4,$$

and hence ξ_2, ξ_3, ξ_4 and similarly every ξ_i. Indeed, if A_1 multiply $\xi_1+\xi_2+\xi_3+\xi_4$ (to take an example) by -1, then A_1 replaces $-\xi_1 + \xi_2 + \xi_3 + \xi_4$ by $-3\xi_1 - \xi_2 - \xi_3 - \xi_4$, the latter not leading to a symbol.

Case (2): $c =$ not-square. There are $p^n - \epsilon$ sets of solutions of

$$\lambda_1^2 + \lambda_2^2 = c, \quad \lambda_1 \neq 0, \quad \lambda_2 \neq 0.$$

Each set of solutions λ_1, λ_2 furnishes only one new dependent set of solutions, viz., $-\lambda_1, -\lambda_2$. Let A_1 multiply $\lambda_1\xi_1 + \lambda_2\xi_2$ by $+1$. If there be a second function $\lambda_1'\xi_1 + \lambda_2'\xi_2$ independent of the former which A_1 multiplies by $+1$, then A_1 leaves ξ_1 and ξ_2 fixed. In the contrary case, A_1 multiplies $p^n - \epsilon - 2$ functions by -1. Among the latter, occur at least two independent functions if $p^n > 5$. Hence CA_1 leaves ξ_1 and ξ_2 fixed. Thus either A_1 or CA_1, say A_2, leaves ξ_1 and ξ_2 fixed. Then A_2 cannot multiply $\lambda_1\xi_1 + \lambda_m\xi_m$ by -1, where $\lambda_1^2 + \dfrac{1}{\mu}\lambda_m^2 = c$, $\lambda_1 \neq 0$, since A_2 would then replace $-\lambda_1\xi_1 + \lambda_m\xi_m$ by $-3\lambda_1\xi_1 - \lambda_m\xi_m$, whereas the latter does not define a symbol. Hence, A_2 leaves ξ_m fixed. Employing $\lambda_i\xi_i + \lambda_m\xi_m$, we see that A_2 leaves ξ_i fixed, and is, therefore, the identity.

If $p^n=5$, and $\mu=\nu$, a not-square, the symbol $\{\lambda_m\xi_m\}$ occurs. Let A_1 denote A or AC according as the former or latter leaves ξ_m fixed. Then $\lambda_i^2+\frac{1}{\nu}\lambda_m^2=c$ has $p^n+\varepsilon=6$ sets of solutions and A_1 cannot multiply $\lambda_i\xi_i+\lambda_m\xi_m$ $(\lambda_i\neq 0)$ by -1. Hence, A_1 leaves every ξ_i fixed.

If $p^n=5$, and $\mu=1$, then 2 is a not-square, so that we may suppose that A_1 leaves $\lambda\xi_1+\lambda\xi_2$ fixed, $2\lambda^2=c$. Since $3(2\lambda)^2\equiv c$ (mod 5), the function $2\lambda(\xi_1+\xi_2+\xi_3)$ leads to a symbol. If A_1 multiplies it by -1, A_1 replaces $2\lambda(-\xi_1-\xi_2+\xi_3)$ by the excluded function $-6\lambda\xi_1-6\lambda\xi_2-2\lambda\xi_3$. Hence, must A_1 multiply the above function by $+1$ and, therefore, leave ξ_3 fixed. Employing $\lambda\xi_i+\lambda\xi_3$, we find that A_1 leaves ξ_i fixed. Hence A_1 is the identity.

If $p^n=3$ and $\mu=\nu=-1$, we may suppose that A_1 leaves ξ_m fixed. Since $c=-1$, there exist only six sets of solutions of

$$\lambda_1^2+\lambda_2^2-\lambda_m^2=-1,$$

yielding the symbols $\{\xi_1+\xi_2\}$, $\{\xi_1-\xi_2\}$ and $\{\xi_m\}$. Hence, if $m=3$, A_1 need not be the identity; it may change the signs of ξ_1 and ξ_2. If, however, $m\geqq 4$, there exist functions

$$\lambda_1\xi_1+\lambda_2\xi_2+\lambda_3\xi_3+\xi_m, \qquad \lambda_1^2=\lambda_2^2=\lambda_3^2=1.$$

If A_1 multiply any one of these by -1, A_1 would replace $\lambda_1\xi_1+\lambda_2\xi_2+\lambda_3\xi_3-\xi_m$ by the excluded function $-\lambda_1\xi_1-\lambda_2\xi_2-\lambda_3\xi_3$. Hence, the multiplier is always $+1$, so that A_1 is the identity.

Case (3): $c=0$, $-1=i^2$, i a mark of the $GF[p^n]$. Among the symbols occur $\{\xi_1\pm i\xi_2\}$. If A multiply $\xi_1+i\xi_2$ by ρ and $\xi_1-i\xi_2$ by σ, then A replaces ξ_1 and ξ_2 by the respective functions

$$\tfrac{1}{2}(\rho+\sigma)\xi_1+\frac{i}{2}(\rho-\sigma)\xi_2, \quad -\frac{i}{2}(\rho-\sigma)\xi_1+\tfrac{1}{2}(\rho+\sigma)\xi_2.$$

By the orthogonal conditions $\rho\sigma=1$ while ξ_j $(j>2)$ is replaced by a function not involving ξ_1 or ξ_2. If $m=3$, ξ_m is replaced by $\pm\xi_m$. If $m>3$, we consider the symbols $\{\xi_1\pm i\xi_j\}$ for $j=3,\ldots\ldots,m-1$ and find that ξ_1 is replaced by a function of ξ_1 and ξ_j. Combining this with the earlier result, ξ_1 is replaced by a function of ξ_1 only and $\rho=\sigma=\pm 1$. If $m>3$, it follows that A is the identity or C. For $m=3$, $\mu=1$, the same result evidently holds. Finally, for $m=3$, $\mu=\nu$, there exist sets of solutions of

$$\lambda_1^2+\lambda_2^2+\frac{1}{\nu}\lambda_m^2=0, \qquad (\lambda_1\neq 0). \tag{33}$$

Let A replace $\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_m\xi_m$ by $\varkappa$ and $-\lambda_1\xi_1 - \lambda_2\xi_2 + \lambda_m\xi_m$ by λ. Then A replaces $2\lambda_m\xi_m$ by $(\varkappa - \lambda)\,\lambda_1\xi_1 + (\varkappa - \lambda)\,\lambda_2\xi_2 + (\varkappa + \lambda)\,\lambda_m\xi_m$. But, for $m = 3$, A was shown to replace ξ_m by $\pm\,\xi_m$. Hence $\varkappa = \lambda = \pm 1$, so that A multiplies $\lambda_1\xi_1 + \lambda_2\xi_2$ by $\pm\,1$. Similarly, A multiplies $\lambda_1\xi_1 - \lambda_2\xi_2$ by $\pm\,1$. Hence, A is either the identity or C.

Case (4): $c = 0$, $-1 =$ not-square. If $\mu = \nu$, we may take $\mu = -1$ and consider the symbols $\{\xi_1 \pm \xi_m\}$, $\{\xi_m \pm \xi_j\}$ for $j = 2, \ldots, m-1$, when, as in Case (3), A is seen to be the identity or C.

If, however, $\mu = 1$, there exist no symbols involving only one or two indices. Consider the $p^{2n} - 1$ sets of solutions of

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0, \qquad (\lambda_1 \neq 0). \tag{33'}$$

Each set furnishes exactly $p^n - 1$ dependent sets obtained by multiplying the given set by the various marks $\neq 0$ of the $GF[p^n]$. Hence, there are $p^n + 1$ groups of dependent sets, those of different groups being independent sets. Hence, there are at least two independent sets

$$(\lambda) \qquad \lambda_{11},\ \lambda_{12},\ \lambda_{13};\ \lambda_{21},\ \lambda_{22},\ \lambda_{23}, \qquad (\text{not every } \lambda_{2j} = \sigma\lambda_{1j})$$

leading to linear functions which A multiplies by the same factor ρ. If the quantities

$$\alpha\lambda_{11} + \beta\lambda_{21}, \quad \alpha\lambda_{12} + \beta\lambda_{22}, \quad \alpha\lambda_{13} + \beta\lambda_{23}$$

be solutions of (33′), then $2\alpha\beta\tau = 0$, $\tau \equiv \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23}$. If $\tau = 0$, every one of these sets of quantities would be a solution of (33′), so that every linear function $\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$ would be multiplied by ρ. Then would A multiply each index by ρ, so that $\rho^2 = 1$, requiring that S be the identity or C [proof as in §15, the right member of (32) being $\rho\lambda_j$]. If, however, $\tau \neq 0$, either $\alpha = 0$ or $\beta = 0$, so that only $2(p^n - 1)$ of the sets of solutions of (33′) are derived linearly from the sets (λ). There remain $(p^n - 1)^2$ sets linearly independent of the sets (λ). If any one of these leads to a linear function $\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$ which A multiplies by ρ, then A multiplies ξ_1, ξ_2 and ξ_3 by $\rho = \pm\,1$ (§15). In the contrary case, they lead to at least two independent functions which A multiplies by the same constant $\varkappa$, since they may be separated into $p^n - 1$ independent groups of sets, none of which lead to a multiplier ρ. Eliminating ξ_3, we obtain a function $a_1\xi_1 + a_2\xi_2$ which A multiplies by $\varkappa$. Likewise from the sets (λ) we obtain a function $b_1\xi_1 + b_2\xi_2$ which A multiplies by $\varkappa$. Since $\varkappa \neq \rho$, these functions are independent, so that A replaces ξ_1 and ξ_2 by functions

of ξ_1 and ξ_2 only. If $m=3$, A replaces ξ_3 by $\pm\xi_3$ in view of the orthogonal conditions, so that A multiplies the functions derived from (λ) by ρ and, therefore, multiplies ξ_1 and ξ_2 each by ρ. If $m>3$, we make the above argument with ξ_2 replaced by ξ_i $(i=4,\ldots,m)$ and see that ξ_1, and then every ξ_j, is multiplied by a contant.

We may combine our results into the theorem:

The orthogonal quotient-group $Q^{(\mu)}_{m,p^n}$, $m>2$, is holoedrically isomorphic with he corresponding substitution group on the symbols $\{\lambda_1\xi_1,+\ldots+\lambda_m\xi_m\}$ in which $\lambda_1^2+\ldots+\lambda^2_{m-1}+\frac{1}{\mu}\lambda^2_m$ is a constant c, the cases $m=3$, $p^n=3$, $\mu=1$ or ν begin exceptional if $c\neq 0$.

For the simple groups defined by the orthogonal groups* and the correspond ing substitution groups, the isomorphism must, of course, be holoedric. By a study of all the invariant subgroups of $Q^{(\mu)}_{m,p^n}$, the above theorem may be otherwise established. For the exceptional case $p^n=3$, $m=3$, the orthogonal groups are of special structure, being then solvable groups.

The abelian linear group, §§17–20.

17. For the substitutions of the abelian group we employ the notation

$$\xi'_i=\sum_{j=1}^{m}(\alpha_{ij}\xi_j+\gamma_{ij}\eta_j),\quad \eta'_i=\sum_{j=1}^{m}(\beta_{ij}\xi_j+\delta_{ij}\eta_j),\qquad (34)$$
$$(i=1,2,\ldots m),$$

the number of indices 2^m being even.

The general abelian group† G is composed of all the substitutions (34) with coefficients in the $GF[p^n]$ satisfying the conditions

$$\sum_{j=1}^{m}(\alpha_{ij}\delta_{ij}-\gamma_{ij}\beta_{ij})=\mu,\quad \sum_{j=1}^{m}(\alpha_{ij}\delta_{kj}-\gamma_{ij}\beta_{kj})=0,\qquad (35)$$

$$\sum_{j=1}^{m}(\alpha_{ij}\gamma_{kj}-\gamma_{ij}\alpha_{kj})=0,\quad \sum_{j=1}^{m}(\beta_{ij}\delta_{kj}-\delta_{ij}\beta_{kj})=0,\qquad (36)$$
$$(i,k=1,\ldots,m;\ i\neq k),$$

since they must leave invariant, up to a factor $\mu\neq 0$, the function

$$\phi\equiv\sum_{i=1}^{m}(x_iY_i=y_iX_i),$$

* American Journal, vol. XXI.

† Dickson, Quarterly Journal, 1897, pp. 169-178. It will be refered to as Q. J.

when operating simultaneously upon the two sets of indices

$$x_i,\ y_i;\quad X_i,\ Y_i. \qquad (i=1,\ldots,m)$$

If $\mu=1$, so that ϕ is an absolute invariant, a subgroup H is defined, which is called the special abelian group.

For $m>1$, the maximal invariant subgroup of H is composed of the identity and the substitution changing the signs of all $2m$ indices, the case $m=2$, $p^n=2$ being an exception, H being then isomorphic with the symmetric group on 6 letters. The quotient group will be designated $A\,(2m, p^n)$. It is *simple* except for $m=1$, $p^n=2$; $m=1$, $p^n=3$; $m=2$, $p^n=2$.

18. Theorem.—*The special abelian group possesses successive generality.*

The relations (35) and (36) do not explicitly include a relation involving only the coefficients α_{1j}, γ_{1j} of the first row of the matrix for (34), but the first relation (35) requires (since $\mu \neq 0$) that the coefficients α_{1j}, γ_{1j} shall not all vanish. But for α_{1j}, γ_{1j} arbitrary marks not all zero, H contains (*Q. J.*, p. 171) a substitution replacing ξ_1 by

$$\omega \equiv \sum_{j=1}^{m} (\alpha_{1j}\xi_j + \gamma_{1j}\eta_j).$$

To prove the theorem for the next step, let α_{1j}, γ_{1j}, β_{1j}, δ_{1j} be an arbitrary set of marks of the $GF[p^n]$ such that

$$\sum_{j=1}^{m} (\alpha_{1j}\delta_{1j} - \gamma_{1j}\beta_{1j}) = 1. \qquad (37)$$

We are to prove that H contains a substitution S which replaces ξ_1 by ω and η_1 by the function

$$\omega_1 \equiv \sum_{j=1}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

By the case already considered, H contains a substitution

$$A:\quad \xi_i' = \sum_{j=1}^{m} (\alpha_{ij}'\xi_j + \gamma_{ij}'\eta_j),\quad \eta_i' = \sum_{j=1}^{m} (\beta_{ij}'\xi_j + \delta_{ij}'\eta_j), \qquad (i=1,\ldots,m)$$

in which $\alpha_{1j}' \equiv \alpha_{1j}$, $\gamma_{1j}' \equiv \gamma_{1j}$, the coefficients subject to relations (35) and (36), when $\mu=1$ and the letters are primed. The inverse of A is

$$A^{-1}:\quad \xi_i' = \sum_{j=1}^{m} (\delta_{ji}'\xi_j - \gamma_{ji}'\eta_j),\quad \eta_i' = \sum_{j=1}^{m} (-\beta_{ji}'\xi_j + \alpha_{ji}'\eta_j). \qquad (i=1,\ldots,m)$$

Since the ratios $\beta_{1j}:\alpha_{1j}$, $\delta_{1j}:\gamma_{1j}$ are not all equal in virtue of (37), there exists a substitution S_1, with coefficients in the $GF[p^n]$ of determinant not zero, which replaces ξ_1 by ω and η_1 by ω_1. Hence the product $A^{-1}S_1 \equiv R$ leaves ξ_1 fixed and replaces η_1 by

$$\rho \equiv \sum_{j=1}^{m} (\rho_j \xi_j + r_j \eta_j),$$

$$\rho_j \equiv \sum_{i=1}^{m} (\beta_{1i}\delta'_{ji} - \delta_{1i}\beta'_{ji}), \quad r_j \equiv \sum_{i=1}^{m} (-\beta_{1i}\gamma'_{ji} + \delta_{1i}\alpha'_{ji}), \qquad (j = 1, \ldots, m).$$

In particular $r_1 = 1$ by (37) since $\alpha'_{1i} = \alpha_{1i}$, $\gamma'_{1i} = \gamma_{1i}$. Since $S_1 = AR$, the product AR replaces ξ_1 by ω and η_1 by ω_1. The required abelian substitution S may therefore be taken to be AR', if there exist in H a substitution R' which, like R, replaces ξ_1 by ξ_1 and η_1 by ρ. Such a choice of R' is possible since (*Q.J.* p. 172) the group H contains a substitution leaving ξ_1 fixed and replacing η_1 by $\sum_{j=1}^{m} (\rho_j \xi_j + r_j \eta_j)$, where $r_1 = 1$ and the remaining ρ_j, η_j are any marks.

The general case of the theorem may be established by induction (compare §5), the method of proof being quite similar to that just employed in proving the first cases of the theorem.

COROLLARY. *The general abelian group G possesses successive generality.*

The substitution of H which replaces ξ_1 by ω, for α_{1j}, γ_{1j} arbitrary marks not all zero, belongs to the larger group G. To prove the next case of the theorem, in which α_{1j}, γ_{1j} β_{1j}, δ_{1j} are any marks satisfying the first condition (35) for $i = 1$, we need only take as the required substitution replacing ξ_1 by ω and η_1 by ω_1 the product ST where S is the substitution of H (determined as above) which replaces ξ_1 by ω and η_1 by $\frac{1}{\mu}\omega_1$ and T is the substitution of G which alters only η_1, multiplying it by μ.

19. An abelian substitution (34) replaces the linear function

$$z \equiv \sum_{i=1}^{m} (x_i \xi_i + y_i \eta_i) \tag{38}$$

by the function $z' \equiv \sum_{i=1}^{m} (x'_i \xi_i + y'_i \eta_i)$, where

$$x'_i \equiv \sum_{j=1}^{m} (\alpha_{ji} x_j + \beta_{ji} y_j), \quad y'_i \equiv \sum_{j=1}^{m} (\gamma_{ji} x_j + \delta_{ji} y_i), \tag{39}$$
$$(i = 1, \ldots, m),$$

Expressed in the matrix notation, these relations become

$$\begin{array}{r|cccccc} & x_1 & y_1 & x_2 & y_2 & \quad x_m & y_m \\ \hline x_1' = & \alpha_{11} & \beta_{11} & \alpha_{21} & \beta_{21} & \cdots\; \alpha_{m1} & \beta_{m1} \\ y_1' = & \gamma_{11} & \delta_{11} & \gamma_{21} & \delta_{21} & \cdots\; \gamma_{m1} & \delta_{m1} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_m' = & \alpha_{1m} & \beta_{1m} & \alpha_{2m} & \beta_{2m} & \cdots\; \alpha_{mm} & \beta_{mm} \\ y_m' = & \gamma_{1m} & \delta_{1m} & \gamma_{2m} & \delta_{2m} & \cdots\; \gamma_{mm} & \delta_{mm} \end{array}$$

the matrix being thus the transposed of the matrix of substitution (34). But if (34) be abelian, then is also the substitution given by the transposed of its matrix. In fact, the conditions that the latter be abelian are precisely the conditions that the inverse of (34) shall be abelian (Q. J., p. 170).

As a first result, we observe that the $p^{2nm}-1$ functions z, obtained by allowing $x_1, \ldots, x_m, y_1, \ldots, y_m$ to run through every set of $2m$ marks not all zero of the $GF[p^n]$, are permuted transitively* by an arbitrary abelian substitution (34). Combining into a single symbol $\{z\}$ the functions $\varkappa z$, where $\varkappa$ runs through the series of marks $\neq 0$ of the $GF[p^n]$, we obtain a set of $(p^{2nm}-1)/(p^n-1)$ symbols which are permuted transitively by an arbitrary abelian substitution. This result corresponds to that of §1 for the general linear group on $2m$ indices. The substitution group is, however, not doubly transitive in the case of the abelian group (see the next paragraph).

The abelian quotient-group $A(2m, p^n)$ may be represented† *as a simply transitive substitution group on $(p^{2nm}-1)/(p^n-1)$ letters.*

To obtain other (less immediate) results, we consider in connection with (39) a second set of functions

$$Z \equiv \sum_{i=1}^{m} (X_i\xi_i + Y_i\eta_i).$$

Upon applying (34), Z is replaced by $Z' \equiv \Sigma\,(X_i'\xi_i + Y_i'\eta_i)$, where X_i' and Y_i' are expressed in terms of X_i, Y_i by formulas similar to (39), i. e., by means

* The relations (39) may be solved for $x_j, y_j\ (j = 1, \ldots, m)$ since the determinant of their coefficients is not zero, being equal to the determinant of (34).

† A special abelian substitution leaving every symbol $\{z\}$ fixed, is either the identity or changes the signs all the indices.

of a matrix which is the transposed of the matrix of substitution (34). It follows that, if (34) be abelian, the following function is an invariant: *

$$\sum_{i=1}^{m}(x_i'Y_i'-y_i'X_i')=\sum_{i=1}^{m}(x_iY_i-y_iX_i).$$

Introducing a positional symbol as in §3, we may state our result in the form:

An abelian substitution permutes amongst themselves the totality of symbols

$$\begin{bmatrix} x_1 & y_1 & x_2 & y_2 & \ldots & x_m & y_m \\ X_1 & Y_1 & X_2 & Y_2 & \ldots & X_m & Y_m \end{bmatrix},\quad \sum_{i=1}^{m}(x_iY_i-y_iX_i)=c,$$

in which c is any constant mark of the $GF[p^n]$.

For the case $c=1$, the above symbols include the following:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Hence, by §18, *the totality of symbols given by $c=1$ are permuted transitively by the special abelian group H and, à fortiori, by the general abelian group G.*

More generally, let c be any mark $\neq 0$. By §18, H contains a substitution S replacing ξ_1 and η_1 by the respective functions

$$\sum_{i=1}^{m}(x_i\xi_i+y_i\eta_i),\quad \sum_{i=1}^{m}(X_ic^{-1}\xi_i+Y_ic^{-1}\eta_i),$$

since, by hypothesis,

$$\sum_{i=1}^{m}(x_iY_ic^{-1}-y_iX_ic^{-1})\equiv c^{1-}\sum_{i=1}^{m}(x_iY_i-y_iX_i)=c^{-1}c=1.$$

Hence, S replaces the first by the second of the following symbols:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & c & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},\quad \begin{bmatrix} x_1 & y_1 & x_2 & y_2 & \ldots & x_m & y_m \\ X_1 & Y_1 & X_2 & Y_2 & \ldots & X_m & Y_m \end{bmatrix}.$$

The totality of symbols defined by a given mark $c \neq 0$ are permuted transitively by the special abelian group.

There remains the case $c=0$. The case $m=1$ may be excluded as trivial. In fact, X_1, Y_1 are then proportional to x_1, y_1 (not both of which are zero), so that the function Z is a constant ($\neq 0$) times the function z, and, therefore, the

* While the result is derived by means of ϕ of §17, the two invariants are not to be confused, the one formed of variables and the other of coefficients.

employment of a symbol of two rows has no advantage over the use of a symbol of one row, a case previously studied. Indeed, for $m=1$, the abelian group is the general binary linear group.

Suppose that $m>1$. In the applications to be made, a symbol is to be excluded as trivial when $X_i=\rho x_i$, $Y_i=\rho x_i$ $(i=1, \ldots, m)$. A linear substitution evidently replaces a trivial symbol by a trivial symbol and a non-trivial symbol by a non-trivial one. Consider the following special and general symbols given by $c=0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 & 0 \end{bmatrix}; \quad \begin{bmatrix} x_1 & y_1 & \ldots & x_m & y_m \\ X_1 & Y_1 & \ldots & X_m & Y_m \end{bmatrix}, \quad \sum_{i=1}^{m}(x_i Y_i - y_i X_i)=0,$$

of which the first is not trivial and the second is assumed to be not trivial. In order that a linear substitution (34) shall replace the former by the latter symbol, following conditions are necessary and sufficient:

$$\alpha_{1i}=x_i, \quad \gamma_{1i}=y_i, \quad \alpha_{2i}=X_i, \quad \gamma_{2i}=Y_i, \quad (i=1, 2, \ldots, m).$$

Hence the first and third rows of the matrix of (34) are determined, and, indeed, so that the abelian condition involving them [formula (36) for $i=1$, $k=2$] is satisfied. To determine the coefficients of the remaining rows so that (34) shall be a special abelian substitution, it suffices, in view of the theorem of §18, to determine β_{1i}, δ_{1i} $(i=1, \ldots, m)$ of the second row, so that

$$\sum_{i=1}^{m}(\delta_{1i}x_i-\beta_{1i}y_i)=1, \quad \sum_{i=1}^{m}(\delta_{1i}X_i-\beta_{1i}Y_i)=0.$$

For $m \geqq 2$, these two equations may always be solved in the field. Suppose that $X_1 \neq 0$, changing if necessary the notation. The second relation then determines δ_{11}. Substituting this value in the first, we obtain the equivalent condition

$$\sum_{i=2}^{m}\delta_{1i}(x_iX_1-X_ix_1)-\sum_{i=1}^{m}\beta_{1i}(y_iX_1-Y_ix_1)=X_1.$$

This relation for $\delta_{12}, \ldots, \delta_{1m}, \beta_{11}, \beta_{12} \ldots, \beta_{1m}$ may always be solved (with, in fact, $2m-2$ of them arbitrary) unless the coefficients on the left all vanish. But this would require that the ratios x_i/X_i, y_i/Y_i $(i=1, \ldots, m)$ should all be equal, since equal to x_1/X_1, which is contrary to hypothesis. We have the result:

The non-trivial symbols given by $c=0$ are permuted transitively by the special abelian group.

20. The number of sets of solutions in the $GF[p^n]$ of

$$\sum_{i=1}^{m}(x_iY_i-y_iX_i)=c \tag{40}$$

is $p^{n(2m-1)}(p^{2mn}-1)$ or $p^{n(2m-1)}(p^{2mn}-1+p^n)$ according as $c\neq 0$ or $c=0$. For $c=1$ this result follows from §18 in connection with Q. J., §§6–7. But the same number of solutions exists for any $c\neq 0$ as for $c=1$. Indeed, if x_i', y_i', X_i', Y_i' be a set of solutions when $c=1$, then x_i', y_i', cX_i', cY_i' gives a set of solutions of (40); two sets of the latter type are identical only when the two corresponding sets of the former type are the same. Subtracting p^n-1 times the number of sets when $c=1$ from p^{4mn}, the total number of sets of $4m$ marks, we obtain the number of sets for $c=0$.

Of the solutions of (40) when $c=0$, the following are to be excluded: the p^{2mn} sets in which $x_i=y_i=0$ while X_i, Y_i are arbitrary; the $p^n(p^{2mn}-1)$ sets in which x_i, y_i are not all zero, while $X_i=\rho x_i$, $Y_i=\rho y_i$ $(i=1,\ldots,m)$, giving the trivial symbols. There remain

$$(p^{2mn}-1)(p^{n(2m-1)}-p^n).$$

For $m=1$, this number vanishes, agreeing with the result in §19.

Combine into a single new symbol $\{\ \ \}$ all the symbols $[\ \ ]$ which are derived by multiplication as follows:

$$\begin{Bmatrix} x_1 & y_1 & \cdots & x_m & y_m \\ X_1 & Y_1 & \cdots & X_m & Y_m \end{Bmatrix} \equiv \begin{bmatrix} \mu x_1 & \mu y_1 & \cdots & \mu x_m & \mu y_m \\ \nu X_1 & \nu Y_1 & \cdots & \nu X_m & \nu Y_m \end{bmatrix}.$$

If $c\neq 0$, then by (40) $\mu\nu=1$, so that the symbols are combined in sets of p^n-1. If $c=0$, they are combined in sets of $(p^n-1)^2$. We obtain the following numbers of new symbols:

$$p^{n(2m-1)}(p^{2mn}-1)/(p^n-1) \quad \text{for } c\neq 0;$$
$$(p^{2mn}-1)(p^{n(2m-1)}-p^n)/(p^n-1)^2 \text{ for } c=0.$$

The latter number is less than the former. Each is greater than the number of letters required by the method, explained at the beginning of §19, except for the excluded case $m=1$.

Whether or not the employment of symbols involving three or more rows

would lead to a representation on a fewer number of letters is not evident. Such an investigation would naturally proceed upon lines similar to those used above.

The first and second hypoabelian groups, §§21–26.

21. Every group of linear homogeneous substitutions on M variables with coefficients in the $GF[2^n]$, which is defined by a quadratic invariant not expressible in the field as a quadratic function of fewer than M variables, is holoedrically isomorphic with one of the three groups (A. J., pp. 222–224, pp. 243–246): the first hypoabelian group G_0, the second hypoabelian group $G_{\lambda'}$, each on $M \equiv 2m$ indices, the special abelian group in the $GF[2^n]$ on $M - 1 \equiv 2m$ indices.

To complete the investigation of the representation of linear groups defined by a quadratic invariant, there remains the case of the hypoabelian groups G_λ, composed of all substitutions (34) in the $GF[2^n]$ which leave formally invariant

$$F_\lambda \equiv \sum_{i=1}^{m} \xi_i \eta_i + \lambda \xi_1^2 + \lambda \eta_1^2 .$$

The group is the first hypoabelian group if $\lambda = 0$; the second hypoabelian group if $\lambda = \lambda'$, where λ' is a particular mark such that

$$\xi_1 \eta_1 + \lambda' \xi_1^2 + \lambda' \eta_1^2$$

is not decomposable into linear factors in $GF[2^n]$.

The conditions for the invariance of F_λ under (34) are the abelian relations (35) and (36) for $\mu = 1$, together with

$$\sum_{j=1}^{m} \alpha_{ij}\gamma_{ij} + \lambda \alpha_{i1}^2 + \lambda \gamma_{i1}^2 = \begin{cases} 0, & (i = 2, \ldots, m) \\ \lambda, & (i = 1) \end{cases}, \tag{41}$$

$$\sum_{j=1}^{m} \beta_{ij}\delta_{ij} + \lambda \beta_{i1}^2 + \lambda \delta_{i1}^2 = \begin{cases} 0, & (i = 2, \ldots, m) \\ \lambda, & (i = 1) \end{cases}. \tag{42}$$

For $p = 2$, the inverse of a special abelian substitution is obtained by replacing $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ by $\delta_{ji}, \beta_{ji}, \gamma_{ji}, \alpha_{ji}$ respectively. From (35), (36), (41), (42) may therefore be obtained an equivalent set of conditions. It follows readily that the transposed of the matrix of a hypoabelian substitution is the matrix of a hypoabelian substitution.

The group G_λ contains a subgroup J_λ of index 2 defined by the additional condition (A. J., p. 231):

$$\sum_{i,j}^{1,\ldots,m} \alpha_{ij}\delta_{ij} + \lambda^2 (\alpha_{11}^2 + \beta_{11}^2 + \gamma_{11}^2 + \delta_{11}^2) = m . \tag{43}$$

If $m > 1$, the group $J_{\lambda'}$ is simple; if $m > 2$, the group J_0 is simple.

22. THEOREM.—*The groups J_λ possess successive generality.*

We are to prove that, for any set of marks, not all zero, of the $GF[2^n]$ satisfying conditions (35), (36), (41), (42) for $i, k = 1, 2, \ldots, t, k \neq i$, where t is any given positive integer $< 2m$, there exists in J_λ a substitution S which replaces ξ_l and η_l by respectively

$$X_l \equiv \sum_{j=1}^{m} (\alpha_{lj}\xi_j + \gamma_{lj}\eta_j), \quad Y_l \equiv \sum_{j=1}^{m} (\beta_{lj}\xi_j + \delta_{lj}\eta_j),$$

for each $l = 1, 2, \ldots, \tau$, in case $t = 2\tau$; but also replaces $\xi_{\tau+1}$ by $X_{\tau+1}$ if $t = 2\tau + 1$.

The theorem has already been proved for $t = 1$. Indeed,* for any set of marks, not all zero, of the $GF[2^n]$ such that

$$\sum_{j=1}^{m} \alpha_{1j}\gamma_{1j} = 0,$$

there exists in J_0 a substitution replacing ξ_1 by X_1. Also (A. J., p. 233), for any set of marks of the $GF[2^n]$ satisfying

$$\sum_{j=1}^{m} \alpha_{1j}\gamma_{1j} + \lambda'\alpha_{11}^2 + \lambda'\gamma_{11}^2 = \lambda',$$

there exists in $J_{\lambda'}$ a substitution which replaces ξ_1 by X_1.

To prove the theorem for $t = 2$, let $\alpha_{1j}, \gamma_{1j}, \beta_{1j}, \delta_{1j}$ be any set of marks of the $GF[2^n]$ such that

$$\sum_{j=1}^{m} (\alpha_{1j}\delta_{1j} - \gamma_{1j}\beta_{1j}) = 1, \quad \sum_{j=1}^{m} \alpha_{1j}\gamma_{1j} + \lambda\alpha_{11}^2 + \lambda\gamma_{11}^2 = \lambda,$$

$$\sum_{j=1}^{m} \beta_{1j}\delta_{1j} + \lambda\beta_{11}^2 + \lambda\delta_{11}^2 = \lambda. \tag{44}$$

By the case $t = 1$ just established, J_λ contains a substitution A and its inverse A^{-1}, both exhibited in §18, such that $\alpha'_{1j} \equiv \alpha_{1j}$, $\gamma'_{1j} \equiv \gamma_{1j}$. Also there exists a linear substitution S_1, with coefficients in the $GF[2^n]$ of determinant not zero, which replaces ξ_1 by X_1 and η_1 by Y_1. Hence, the product $A^{-1}S_1 \equiv R$ leaves ξ_1 fixed and replaces η_1 by the function ρ of §18, where, in particular, $r_1 = 1$. We pro-

* Bull. Amer. Math. Soc., vol. IV (1898), p. 498.

ceed to prove that* (mod 2)

$$\sum_{j=1}^{m} \rho_j r_j + \lambda \rho_1^2 = \lambda - \lambda r_1^2 = 0. \tag{45}$$

By a simple arrangement of the summation indices, we have

$$\begin{aligned}\sum_{j=1}^{m} \rho_j r_j = & \sum_{i,\,k}^{1,\ldots,m} \beta_{1i}\delta_{1k} \sum_{j=1}^{m} (\alpha'_{jk}\delta'_{ji} + \beta'_{jk}\gamma'_{ji}) + \sum_{i=1}^{m} \beta_{1i}^2 \sum_{j} \delta'_{ji}\gamma'_{ji} + \sum_{i=1}^{m} \delta_{1i}^2 \sum_{j} \alpha'_{ji}\beta'_{ji} \\ & + \sum_{i<k}^{1,\ldots,m} \beta_{1i}\beta_{1k} \sum_{j=1}^{m} (\delta'_{ji}\gamma'_{jk} + \delta'_{jk}\gamma'_{ji}) + \sum_{i<k}^{1,\ldots,m} \delta_{1i}\delta_{1k} \sum_{j=1}^{m} (\alpha'_{ji}\beta'_{jk} + \alpha'_{jk}\beta'_{ji}) \\ = & \sum_{i=1}^{m} \beta_{1i}\delta_{1i} + \sum_{i=2}^{m} \beta_{1i}^2 (\lambda\delta'^2_{1i} + \lambda\gamma'^2_{1i}) + \beta_{11}^2 (\lambda + \lambda\delta'^2_{11} + \lambda\gamma'^2_{11}) \\ & + \sum_{i=2}^{m} \delta_{1i}^2 (\lambda\alpha'^2_{1i} + \lambda\beta'^2_{1i}) + \delta_{11}^2 (\lambda + \lambda\alpha'^2_{11} + \lambda\beta'^2_{11}),\end{aligned}$$

since the coefficients of $\beta_{1i}\beta_{1k}$, $\delta_{1i}\delta_{1k}$ and $\beta_{1i}\delta_{1k}$ (the latter if $i \neq k$) vanish, while that of $\beta_{1i}\delta_{1k}$ ($i = k$) is unity, in view of the abelian conditions (35) and (36) when written for the inverse of (34) [see §21]. The other reductions were made by using the relations corresponding to (41) and (42) for the inverse. Adding the term

$$\lambda \rho_1^2 \equiv \lambda \sum_{i=1}^{m} (\beta_{1i}^2 \delta'^2_{1i} + \delta_{1i}^2 \beta'^2_{1i}) \quad (\text{mod } 2)$$

and setting $\alpha'_{1j} = \alpha_{1j}$, $\gamma'_{1j} = \gamma_{1j}$, we obtain the sum

$$\sum_{i=1}^{m} \beta_{1i}\delta_{1i} + \lambda\beta_{11}^2 + \lambda\delta_{11}^2 + \lambda \sum_{i=1}^{m} (\beta_{1i}^2\gamma_{1i}^2 + \delta_{1i}^2\alpha_{1i}^2),$$

which $\equiv 0$ (mod 2) in view of the first and third relations (44). By the origin of R and S_1, the product $S_1 = AR$ replaces ξ_1 by X_1 and η_1 by Y_1.

Inversely, if R' be any substitution which replaces ξ_1 by ξ_1 and η_1 by ρ, the product AR' will replace ξ_1 and η_1 by X_1 and Y_1. Hence, if R' can be chosen to belong to J_λ, the required hypoabelian substitution S may be taken to be AR'. In view of relation (45), there exists in J_λ, for $\lambda = 0$, such a substitution R' (Bulletin, l. c., p. 498). For $\lambda = \lambda'$, there exists in J_λ such a substitution R' (A. J., p. 234).

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* **A shorter, but indirect, proof is sketched in §26. The third relation (44) would form the basis, its left member being identified with F^λ of §21.**

By a similar method we may establish the general case of the theorem by one-stage induction [compare §5].

Since the relation (43), distinguishing the subgroup J_λ from the total hypoabelian group G_λ, involves the coefficients of all the rows of the matrix, it does not affect the cases of the theorem for which $t < 2m$. But for $t = 2m$, the theorem is evident from the definition of the group. Hence, the above proof shows also that *the groups G_λ possess successive generality.*

23. In the definition of $G_{\lambda'}$, the indices ξ_1, η_1 play a special rôle, and hence the coefficients α_{1j}, β_{1j}, γ_{1j}, δ_{1j}, α_{j1}, β_{j1}, γ_{j1}, δ_{j1}, enter in a special way into the relations (41), (42), and the analogous relations derived from the inverse substitutions.

By a change of notation, ξ_m and η_m may be given the special rôle, and hence also the coefficients of the last row and last column of the matrix. From this new standpoint the groups possess successive generality, the first case of the theorem being known (A. J., pp. 227–8).

24. For any one of the $N^{(\lambda)}_{m,n}$ sets of solutions (not all zero) in the $GF[2^n]$ of

$$\sum_{j=1}^{m} a_{1j}c_{1j} + \lambda a_{11}^2 + \lambda c_{11}^2 = \lambda, \tag{46}$$

these exists (§22) in J_λ a substitution T replacing ξ_1 by $\sum_{j=1}^{m}(a_{1j}\xi_j + c_{1j}\eta_j)$. If therefore S be any substitution of J_λ, the product ST will belong to J_λ and hence will replace ξ_1 by a function $\Sigma\,(a'_{1j}\xi_j + c'_{1j}\eta_j)$, whose coefficients satisfy (46). All such linear functions are therefore permuted transitively by J_λ. For $\lambda = 0$, ξ_i, $\eta_i\,(i = 1, \ldots, m)$ occur among these linear functions; a substitution of J_0 which leaves them all fixed is the identity. For $\lambda = \lambda'$, the functions

$$\xi_1 + \xi_2,\ \eta_1 + \xi_2,\ \xi_i,\ \eta_i \qquad (i = 2, \ldots, m)$$

occur among the above linear functions defined by (46). A substitution of J'_λ which leaves them fixed is evidently the identity. Similar remarks hold for the groups G_λ. We may therefore state the theorem:

If $m > 1$, the groups J_λ and G_λ may be represented as transitive substitution groups on $N^{(\lambda)}_{m,n}$ letters.

The number of sets of solutions of (46) is known (*A.J.*, p. 230, p. 235):

$$N^{(0)}_{m,n} = (2^{nm} - 1)(2^{n(m-1)} + 1), \quad N^{(\lambda')}_{m,n} = (2^{nm} + 1)\,2^{n(m-1)}.$$

In the evaluation for $\lambda = 0$, the set of solutions $a_{1j} = c_{1j} = 0$ $(j = 1, \ldots, m)$ has been excluded. For $\lambda = 0$, (46) is homogeneous, so that we may group the linear functions into sets of $2^n - 1$ each, those of one set differing only by a constant factor. The resulting symbols

$$\{a_{11}\xi_1 + c_{11}\eta_1 + \ldots + a_{1m}\xi_m + c_{1m}\eta_m\},$$

are permuted transitively by J_0. A substitution S leaving every symbol fixed multiplies each index by a constant* ρ; if S belong to J_0, the abelian condition (35), for $\mu = 1$, gives $\rho^2 = 1$, whence $\rho \equiv 1$ modulo 2.

If $m > 1$, *the groups* J_0 *and* G_0 *may be represented as transitive substitution groups on* $(2^{nm} - 1)(2^{n(m-1)} + 1)/(2^n - 1)$ *letters.*

25. We next treat $J_{\lambda'}$ from the standpoint indicated in §23. For any set of solutions (not all zero) in the $GF[2^n]$ of

$$\sum_{j=1}^{m} a_{mj}c_{mj} + \lambda' a_{m1}^2 + \lambda' c_{m1}^2 = 0, \tag{47}$$

there exists (A. J., p. 227) in $J_{\lambda'}$ a substitution which replaces ξ_m by

$$\sum_{j=1}^{m} (a_{mj}\xi_j + c_{mj}\eta_j).$$

The number of such sets of solutions is (A. J., p. 230)

$$(2^{nm} + 1)(2^{n(m-1)} - 1).$$

Proceeding as in §24, we obtain the theorem†:

If $m > 1$, *the groups* $J_{\lambda'}$ *and* $G_{\lambda'}$ *may be represented as transitive substitution groups on* $(2^{nm} + 1)(2^{n(m-1)} - 1)/(2^n - 1)$ *letters.*

This number is less than the number $N_{m,n}^{(\lambda')}$ of §24.

26. To obtain a representation of J_0 upon a smaller number of letters than that given by the last theorem of §24, we employ the general method used for

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* Consider in connection with the symbols $\{\xi_1\}$, $\{\eta_1\}$, $\{\xi_2\}$, etc., also the symbols $\{\xi_1 + \xi_2\}$, $\{\xi_1 + \eta_2\}$, etc. The various multipliers are thus seen to be equal.

† A substitution of $J_{\lambda'}$ is the identity if it leave fixed the symbols

$$\{\xi_i\}, \{\eta_i\}, \{\xi_1 + \lambda'^{\frac{1}{2}}\xi_i + \lambda'^{\frac{1}{2}}\eta_i\}, \{\eta_1 + \lambda'^{\frac{1}{2}}\xi_i + \lambda'^{\frac{1}{2}}\eta_i\} \qquad (i = 2, \ldots, m),$$

since ξ_i and η_i are multiplied by the same constant [unity by (35)] and since a like result holds for ξ_1 and η_1.

the orthogonal groups (§8, seq). The functions $a_{11}\xi_1 + c_{11}\eta_1 + \ldots + a_{1m}\xi_m + c_{1m}\eta_m$ in which

$$\sum_{j=1}^{m} (a_{1j}c_{1j} + \lambda a_{11}^2 + \lambda c_{11}^2 = c = \text{constant} \tag{48}$$

are permuted amongst themselves by the substitutions of G_λ. The simplest proof may be based upon §3, noting first that the left member of (48) has the form of the defining invariant F_λ of the group G_λ (§21) and secondly that the transposition of the matrix of a hypoabelian substitution leads to a hypoabelian substitution. A proof following the method of §8 results from the computation of (45) made in §22.

The case $c = 0$ has been treated in §§24–25. For different values of $c \neq 0$, relation (48) has evidently the same number of sets of solutions, since every mark of the $GF\,[2^n]$ is a square. For $\lambda = \lambda'$, we take $c = \lambda'$ and have the case treated in §24.

There remains the case $\lambda = 0$, $c \neq 0$, when we take $c = 1$. From the known number of solutions of (46), for $\lambda = 0$, we derive at once the result that the equation

$$\sum_{j=1}^{m} a_{1j}\, c_{1j} = 1 \tag{48'}$$

has in the $GF\,[2^n]$ the following number of sets of solutions:

$$(2^{nm} - 1)(2^{nm} - 2^{n(m-1)})/(2^n - 1) \equiv (2^{nm} - 1)\, 2^{n(m-1)}.$$

Among the functions $a_{11}\xi_1 + c_{11}\eta_1 + \ldots$ defined by (48′) occur (if $m > 1$),

$$\xi_1 + \eta_1, \quad \xi_1 + \eta_1 + \xi_i, \quad \xi_1 + \eta_1 + \eta_i, \quad \xi_2 + \eta_2 + \xi_1,$$
$$(i = 2, \ldots, m).$$

A substitution on the ξ_i, η_i, which leaves fixed all of these functions, will leave fixed every index and, therefore, be the identity. The isomorphism of G_λ with the substitution group is therefore holoedric. We may state the result:

If $m > 1$, the groups J_0 and G_0 may be represented as transitive substitution groups on $(2^{nm} - 1)\, 2^{n(m-1)}$ letters.

27. Comparing the number of letters required by the final theorem of §24 for the representation of J_0 and G_0 with the number of letters required by §26,

we find that the former is less than the latter if $n > 1$ but is greater if $n \quad 1$. For $n = 1$, these numbers are $2^{2m-1} + 2^{m-1} - 1$ and $2^{2m-1} - 2^{m-1}$ respectively.* For this case ($n=1$) our results for the representation of the first hypoabelian group as transitive substitution groups agree with the results of the lengthy investigation by M. Jordan upon the groups of Steiner.† As we are now in a position to pass directly from any hypoabelian substitution to the corresponding substitution of the isomorphic substitution group, the results of M. Jordan may be proven quite simply. This investigation and the generalization to arbitrary n will be deferred to a later paper.

THE UNIVERSITY OF CHICAGO, *February*, 1901.

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* By §25, J_λ' for $n=1$ may be represented transitively on $2^{2m-1} - 2^{m-1} - 1$ letters. Adjoining the symbol $\{ 0 \}$, we obtain an intransitive representation upon the same number of letters as employed for the group J_0.

† "Traité des substitutions," pp. 229–249. See particularly Nos. 318, 347. In the defining congruence No. 318, line 5, the right member should read 1 instead of 0.

A Class of Number-Systems in Six Units.

By G. P. Starkweather.

§1.

It has been shown by Scheffers* that complex number systems in n units can be divided into two distinct classes. In any system of the first class, called, after its best-known representative, the quaternion class, there exist three quantities, e_1, e_2, e_3, between which and the modulus, or idemfactor, no linear relation exists, such that

$$\left.\begin{aligned} e_1e_2 - e_2e_1 &= 2e_3, \\ e_2e_3 - e_3e_2 &= 2e_1, \\ e_3e_1 - e_1e_3 &= 2e_2. \end{aligned}\right\} \tag{1}$$

For every number-system of the second class, to which the name non-quaternion is given, it is possible to choose as units quantities

$$u_1, \ldots . \, u_r, \quad \eta_1, \ldots . \, \eta_s,$$

which have the following multiplicative properties: u_iu_j, and u_ju_i, $j \not> i$, are linear functions of $u_1, \ldots . \, u_{j-1}$; $\eta_i^2 = \eta_i$; $\eta_i\eta_k = 0$, $i \neq k$; η_iu_k is zero except for one value of i, say λ_k, when it equals u_k, and similarly, $u_k\eta_i$ is zero except for one value of i, say μ_k, when it equals u_k. If $\mu_k \neq \lambda_k$, the unit u_k is said to be skew, otherwise it is called even. This form is called the regular form, and no quaternion system can be put in it, nor does any non-quaternion system contain quantities satisfying the equations (1).

If we consider now non-quaternion systems without skew units, if there be more than one of the quantities η, the system can be reduced to a sum of systems containing each only one η.† Therefore, we may assume that in the systems

* "Complexe Zahlensysteme," Mathematische Annalen, XXXIX, pp. 306, 310. † Ibid., p. 328.

considered there are $(n-1)$ of the units u and only one η, which is the modulus. These will be called simple systems. Any number

$$x = a_1 u_1 + \ldots + a_{n-1} u_{n-1} + \xi\eta$$

(where $a_1, \ldots a_{n-1}, \xi$ are ordinary complex quantities) satisfies the equation*

$$(x - \xi\eta)^v = 0,$$

where v is a positive integer not greater than n. This is the characteristic equation. If $v = n - \delta$, δ may be called the deficiency of the system.

In a preceding paper† the writer has considered this class of systems, and showed that by a proper selection of new units the system would be reduced to a form having multiplicative properties which, when δ equals two, or when δ is small in comparison with $n - \delta - 1$, are simpler than those of Scheffers' regular form. The case $\delta = 2$ was then taken up in detail, and certain general properties deduced, and finally, a determination was made of all such systems which are linearly independent for the case $n > 6$ and the parameters reduced to the smallest possible number. The case $n < 3$ cannot occur, the cases $n = 3, 4, 5$ have already been considered by Scheffers by other methods, while the case $n = 6$ presented especial difficulties, to overcome which the writer has not had the time until the present paper.

The problem then is, *to determine all the linearly independent simple non-quaternion number-systems containing no skew units, which can be formed from six units, and which are of deficiency two, and to reduce the parameters to the smallest number.*

§2.

As was proved in the preceding paper,‡ the system can be put into the following form:

* Ibid., p. 316.

† American Journal of Mathematics, vol. XXI, No. 4, p. 369.

‡ L. c., p. 380.

	w_1	w_2	w_3	τ_1	τ_2	η
w_1	0	0	0	0	0	w_1
w_2	0	0	w_1	0	$\frac{a}{2}(3c+d)w_1$	w_2
w_3	0	w_1	w_2	aw_1	$c\tau_1 + ew_1 + \frac{a}{2}(c+d)w_2$	w_3
τ_1	0	0	$-aw_1$	bw_1	fw_1	τ_1
τ_2	0	$-\frac{a}{2}(3d+c)w_1$	$d\tau_1 - ew_1 - \frac{a}{2}(c+d)w_2$	gw_1	$j\tau_1 + hw_1 + iw_2$	τ_2
η	w_1	w_2	w_3	τ_1	τ_2	η

Application of the associative law, and also of the fact that any number x formed from the first five units must satisfy the characteristic equation $x^4 = 0$, yields the following relations, necessary and sufficient, between the parameters $a, b, c, \ldots . j$:

$$jb = 0, \tag{2}$$
$$cb = 0, \tag{3}$$
$$db = 0, \tag{4}$$
$$i + ja = cf + \frac{a^2}{4}(c+d)(3c+d), \tag{5}$$
$$i - ja = dg + \frac{a^2}{4}(c+d)(3d+c), \tag{6}$$
$$df - cg = \frac{a^2}{2}(c+d)(c-d) \tag{7}$$
$$j(f-g) + 2ai(c+d) = 0. \tag{8}$$

These give rise to the following cases:

$b \neq 0.$ Then $c = d = i = j = 0.$ (I)

$b = a = 0,\ c \neq 0.$ Then, first, $f^2 = g^2$, yielding

$$f = \quad g = i = 0. \tag{II}$$

$$f = \quad g \neq 0,\ \ c = d,\quad\ i = cf. \tag{III}$$

$$f = -g \neq 0,\ \ c = -d,\ \ i = cf,\ \ j = 0. \tag{IV}$$

$b = a = c = 0,\ \ d \neq 0.$ Then $f = g = i = 0$. (II′)

$b = a = c = d = 0,\ \ f \neq g.$ Then $i = j = 0$. (V)

$b = a = c = d = 0,\ \ f = g.$ Then $i = 0$. (VI)

$b = 0,\ \ a \neq 0,\ \ c = -d \neq 0.$ Then $f = -g,\ \ i = cf,\ j = 0$. (VII)

$b = 0,\ \ a \neq 0,\ \ c = d = 0.$ Then $i = j = 0$. (VIII)

$b = 0,\ \ a \neq 0,\ \ c \neq -d.$ Then either

$$\left\{\begin{array}{lll} c = -\dfrac{a+f}{a^2}, & d = \dfrac{3a+f}{a^2}, & g = -4a - f, \\ i = -\dfrac{(f+2a)^2}{a^2}, & j = \dfrac{2(f+2a)}{a^2}, & f \neq -a,\ -3a, \end{array}\right\} \tag{IX}$$

or

$$\left\{\begin{array}{l} c = \dfrac{m+a}{a(m-a)},\quad d = \dfrac{m-3a}{a(m-a)},\quad f = -\dfrac{m^2+3a^2}{4a}, \\ g = -\dfrac{m^2 - 4\,ma + 7a^2}{4a},\quad i = -\dfrac{(m-a)^2}{4a^2}, \\ j = -\dfrac{m-a}{a^2},\quad m \neq a. \end{array}\right\} \tag{X}$$

II′ is included among the reciprocals of II. All these cases follow easily from equations (2), ... (8), except IX and X. We proceed to consider these more closely.

We have by hypothesis $b = 0,\ a \neq 0,\ c \neq -d$. By replacing τ_2 by the new unit $\tau_2' = \dfrac{2}{a(c+d)}\tau_2$ we obtain a new form of the same type with c and d replaced by $c' = \dfrac{2c}{a(c+d)}$ and $d' = \dfrac{2d}{a(c+d)}$ respectively, whence $a\,(c' + d') = 2$. This being of the same typical form, the equations (2), ... (8) will hold written with primes. Dropping the primes, we can thus assume $a\,(c+d) = 2$, and the equations become

$$i + ja = cf + ac + 1, \tag{5′}$$

$$i - ja = \left(\frac{2}{a} - c\right) g + 3 - ac, \tag{6′}$$

$$\frac{2}{a} f - cf - cg = 2ac - 2, \tag{7′}$$

$$j\,(f-g) + 4i = 0. \tag{8′}$$

From (7′)

$$c = \frac{2 + \frac{2}{a} f}{2a + f + g}, \tag{9}$$

unless

$$2a + f + g = 0. \tag{9′}$$

But in that case we have

$$i - ja = -\frac{2f}{a} - 1 + cf + ac, \tag{6″}$$

$$\frac{2}{a} f = -2. \tag{7″}$$

From these, with (5′), $f = -a$, $i = 1$, $j = 0$, which contradict (8′). Hence (9′) is impossible, and accordingly, (9) is always true. Substituting in (5′) and (6′) we obtain

$$i = \frac{g^2 + 3ag + 3af + f^2 + 4a^2}{a(2a + f + g)}, \tag{10}$$

$$j = \frac{f - g}{a^2} \tag{11}$$

Substituting in (8′) there results

$$g^3 + 6ag^2 - g^2 f + 12a^2 g - 4agf + 12a^2 f - f^2 g + 6af^2 + f^3 + 16a^3 = 0,$$

whence

$$g = -4a - f, \tag{12}$$

$$g = f - a \pm \sqrt{-4af - 3a^2}. \tag{13}$$

Both of these, with the corresponding values of c, i and j obtained from (9), (10) and (11), satisfy (5′), ... (8′). The possibility of (9′) being true must, however, be excluded. This cannot occur when equation (12) is taken, as it makes a equal to zero, contrary to hypothesis. When equation (13) is used, it necessitates that $f = -a$ shall be excluded when the upper sign of the radical is taken.

When $f = -a$, (13), using the lower sign gives the same value of g, hence of c, i and j, that (12) does. The same is true when $f = -3a$, using the upper sign of 13. Hence it can be assumed in (12) that $f \neq -a, -3a$. This, remembering that $a(c + d) = 2$, gives case IX, p. 381.

Considering now (13), as the radical is awkward, introduce the new parameter $m = \sqrt{-4af - 3a^2}$, whence $f = -\dfrac{m^2 + 3a^2}{4a}$.

As with f equal to $-a$ we must exclude the positive sign of the radical,

$m \neq a$. As m takes on all other values, f takes on all values, and conversely; also the equation

$$g = -\frac{m^2 - 4ma + 7a^2}{4a}$$

gives all values of g yielded by (13) for the corresponding values of f. Hence m can be used as a parameter in place of f, with $m \neq a$, and from (9), (10), (11) and the fact that $a(c+d) = 2$, we obtain case X.

By replacing τ_2 by $\tau_2' = \tau_2 + \alpha\tau_1$, α being properly chosen, h can be made zero in I and III, and e can be made zero in VII, VIII, IX and X. In IV e can be reduced to zero by replacing τ_1 by $\tau_1' = \tau_1 + \alpha w_1$, and then h can be made zero by replacing w_3 and w_2 by $w_3' = w_3 + \alpha w_2$ and $w_2' = w_2 + 2\alpha w_1$ respectively. So simplified, the forms on p. 381 will be called typical forms. A partial list of these forms was given, in the preceding paper.* Of the six there given, I II, IV, V and VI are respectively identical with those here numbered I, VIII II, V and VI, while III there is included in VII here.

A nilfactor will be defined as a quantity ν, different from zero, such that $\nu x = x\nu = 0$ for all values of x, x being a number of the system. An alternate will be defined as a quantity α, different from zero, such that $\alpha x = -x\alpha$ for all values of x. A nilfactor is thus also an alternate. Such quantities evidently cannot exist in a complete system, that is, a system containing a modulus. The system given in the table on p. 380, with η deleted, is incomplete, since it contains the nilfactor w_1. By actual trial in each of the given typical forms, the following theorems can be demonstrated:

I. *The incomplete typical forms possess no nilfactors except linear functions of w_1 and such τ's as are themselves nilfactors.*

II. *The incomplete typical forms possess no alternates except linear functions of w_1 and such τ's as are themselves alternates.*

§3.

We next proceed to determine the inequivalent systems and reduce the parameters as far as possible. Suppose two systems, $w_1, w_2, w_3, \tau_1, \tau_2, \eta$, and $w_1', w_2', w_3', \tau_1', \tau_2', \eta'$ are equivalent. Evidently $\eta = \eta'$. Consider any unit u' of the second system different from η'. Then

$$u' = a_1 w_1 + a_2 w_2 + a_3 w_3 + b_1 \tau_1 + b_2 \tau_2 + c\eta.$$

* L. c., p. 381.

But every quantity x in the incomplete system $w'_1, w'_2, w'_3, \tau'_1, \tau'_2$ must satisfy the characteristic equation $x^4=0$. In order that $u'^4=0$, it is necessary that c equal zero. Hence the incomplete systems $w_1, w_2, w_3, \tau_1, \tau_2$ and $w'_1, w'_2, w'_3, \tau'_1, \tau'_2$ are equivalent, and we can, therefore, drop η out entirely. The transformations for w'_1 and w'_2 need not be given, for they follow as powers of w'_3.

By interchanging the τ's, I goes into V when $e=0$, $f\neq g$, into VI when $e=0$, $f=g$, and into VIII when $e\neq 0$. So I drops out.

In II, we can make d zero if $h=0$, $j\neq 0$, by the transformation $w'_3 = w_3+\alpha\tau_2$. This changes the value of c, which, by hypothesis, is not equal to zero. But should it reduce to zero by the transformation, the system would come into form VI. When $c=-d$, e can be made zero by the transformation $\tau'_1=\tau_1+\alpha w_1$.

In V, when $f\neq -g$, h can be made equal to zero by the transformation $\tau'_2=\tau_2+\alpha\tau_1$, and when $f=-g$, we can make e equal to zero by $w'_3=w_3+\alpha\tau_1$.

In VI we can reduce h to zero when $f\neq 0$ by the transformation $\tau'_2 = \tau_2+\alpha\tau_1$, and, when $f=0, j\neq 0$, the same can be done by the transformation $\tau'_1=\tau_1+\alpha w_1$.

VIII goes into either V or VI when $h=0$ by interchanging the τ's.

Transformations $w'_3=xw_3$, $\tau'_1=y\tau_1$, $\tau'_2=z\tau_2$ enable the following parameters to be reduced to unity, provided they are not zero: In II, c and any two of e, h and j. In II′, d and any two of e, h and j. In III, c, e and f. In IV, c and f. In V, e, h and one of f and g. In VI, any three of e, f, h and j. In VII, a, c and h. In VIII, a, h and one of f and g. In IX, a and h. In X, a and h.

The preceding facts yield the most of the following subdivisions and reductions of the typical forms. Some of the subdivisions are made, not from the preceding, but for reasons explained on pp. 386, 387. The subcases of II′ are reciprocals of the corresponding cases under II:

II.	(A)	$c=1$	$h=0$	$j=0$	$e=0$	$d^2\neq 1$,
	(B)	$c=1$	$h=0$	$j=0$	$e=0$	$d=1$,
	(C)	$c=1$	$h=0$	$j=0$	$e=0$	$d=-1$,
	(D)	$c=1$	$h=0$	$j=0$	$e=1$	$d^2\neq 1$,
	(E)	$c=1$	$h=0$	$j=0$	$e=1$	$d=1$,
	(F)	$c=1$	$h=0$	$j=1$	$e=0$	$d=0$,
	(G)	$c=1$	$h=0$	$j=1$	$e=1$	$d=0$,

	(H)	$c=1$	$h=1$	$j=0$	$e=0$	$d^2\neq 1$,
	(I)	$c=1$	$h=1$	$j=0$	$e=0$	$d=1$,
	(J)	$c=1$	$h=1$	$j=0$	$e=0$	$d=-1$,
	(K)	$c=1$	$h=1$	$j=1$	$e=0$	$d\neq 1$,
	(L)	$c=1$	$h=1$	$j=1$	$e=0$	$d=1$,
	(M)	$c=1$	$h=1$	$j=j$	$e=1$	$d^2\neq 1$.
	(N)	$c=1$	$h=1$	$j=j$	$e=1$	$d=1$.
II′.	(A)	$d=1$	$h=0$	$j=0$	$e=0$,	
	(D)	$d=1$	$h=0$	$j=0$	$e=1$,	
	(F)	$d=1$	$h=0$	$j=1$	$e=0$,	
	(G)	$d=1$	$h=0$	$j=1$	$e=1$,	
	(H)	$d=1$	$h=1$	$j=0$	$e=0$,	
	(K)	$d=1$	$h=1$	$j=1$	$e=0$,	
	(M)	$d=1$	$h=1$	$j=j$	$e=1$.	
III.	(A)	$c=1$	$f=1$	$e=1$,		
	(B)	$c=1$	$f=1$	$e=0$.		
IV.	(A)	$c=1$	$f=1$.			
V.	(A)	$f=1$	$g^2\neq 1$	$h=0$	$e=0$,	
	(A′)	$f=0$	$g=1$	$h=0$	$e=0$,	
	(B)	$f=1$	$g^2\neq 1$	$h=0$	$e=1$,	
	(B′)	$f=0$	$g=1$	$h=0$	$e=1$,	
	(C)	$f=1$	$g=-1$	$h=0$	$e=0$,	
	(D)	$f=1$	$g=-1$	$h=1$	$e=0$.	
VI.	(A)	$f=1$	$j=1$	$e=1$	$h=0$,	
	(B)	$f=1$	$j=0$	$e=1$	$h=0$,	
	(C)	$f=1$	$j=1$	$e=0$	$h=0$,	
	(D)	$f=1$	$j=0$	$e=0$	$h=0$,	
	(E)	$f=0$	$j=1$	$e=1$	$h=0$,	
	(F)	$f=0$	$j=1$	$e=0$	$h=0$,	
	(G)	$f=0$	$j=0$	$e=1$	$h=1$,	
	(H)	$f=0$	$j=0$	$e=1$	$h=0$,	
	(I)	$f=0$	$j=0$	$e=0$	$h=1$,	
	(J)	$f=0$	$j=0$	$e=0$	$h=0$.	

VII.	(A)	$a=1$	$c=$	1	$h=1$	$f\neq 0$.
	(B)	$a=1$	$c=$	1	$h=1$	$f=0$,
	(C)	$a=1$	$c=$	1	$h=0$	$f\neq 0$,
	(D)	$a=1$	$c=$	1	$h=0$	$f=0$.
VIII.	(A)	$a=1$	$h=$	1	$f=1$	$g\neq -1$,
	(B)	$a=1$	$h=$	1	$f=1$	$g=-1$,
	(A′)	$a=1$	$h=$	1	$f=0$	$g=\ \ 1$,
	(C)	$a=1$	$h=$	1	$f=0$	$g=\ \ 0$.
IX.	(A)	$a=1$	$h=$	1	$f\neq -1, -3$,	
	(B)	$a=1$	$h=$	0	$f\neq -1, -3$.	
X.	(A)	$a=1$	$h=$	1	$m\neq 1,\ 1\pm 2\sqrt{-1}$,	
	(B)	$a=1$	$h=$	1	$m=\ \ 1\pm 2\sqrt{-1}$,	
	(C)	$a=1$	$h=$	0	$m\neq 1,\ 1\pm 2\sqrt{-1}$,	
	(D)	$a=1$	$h=$	0	$m=\ \ 1\pm 2\sqrt{-1}$.	

Here are 54 cases, and to test them for equivalence might require 1431 applications of the general linear transformation. The process is greatly reduced by the following considerations, remembering that throughout we need only consider the incomplete systems, η being deleted.

First the systems can be divided according as they are commutative or non-commutative. Second, since the number of linearly independent nilfactors is evidently a characteristic of the incomplete system, by the theorem on p. 383 these two groups can be divided according as none, one, or two of the τ's are nilfactors, the last case of which can occur, of course, only in the commutative class. Third, since the number of linearly independent alternates is evidently a characteristic of the incomplete system, the subgroups of the non-commutative class can be subdivided according as none, one, or two of the τ's are alternates. In the commutative classes alternates must be also nilfactors, hence it yields no new subdivisions for them. These considerations separate the systems into eight distinct classes.

Next suppose two systems $w_1, w_2, w_3, \tau_1, \tau_2$ and $w'_1, w'_2, w'_3, \tau'_1, \tau'_2$ are equivalent. Then w'_3 is linear in $w_1, w_2, w_3, \tau_1, \tau_2$. Hence from the general table on p. 380, w'_2, which equals w'^2_3, is linear in w_1, w_2, τ_1. Therefore w'_1, which equals $w'_3 w'_2$, is linear in w_1, for products of w_1, w_2 and τ_1,

with any units of the incomplete system contain only w_1. Hence w_1 and w_1', both nilfactors, have the relation $w_1' = cw_1$. Now in the preceding paper* the writer has proved the following theorem:

If two incomplete systems are equivalent, and nilfactors ν, ν', having the relation $\nu' = c\nu$, are units in the respective systems, then if ν and ν' be deleted in each system the resulting systems are equivalent.

Therefore in our two systems, if w_1 and w_1' be deleted, the resulting systems in four units (excluding η) are equivalent. But the new systems will be in the typical $w - \tau$ forms given in the preceding paper† for $n = 5$, with w_2 taking the place of w_1, and theorems given on p. 383 hold for these. Hence we can subdivide each of the eight classes above according to the commutative, nilfactive, and alternate properties of the τ's with w_1 deleted. This gives a total of eighteen distinct classes, and each system need be tested for equivalence with only those of its own class. This will require at most 143 applications of the general linear transformation, in fact, far less.

A number of special cases on pp. 384, 385, 386 are necessary for these various subdivisions, as was mentioned on p. 384.

The systems in the different classes follow below. In designating the classes c stands for commutative, n for non-commutative, the first number gives the number of τ's which are nilfactors, the second the number of τ's which are alternates, but not nilfactors. This is not given in the commutative systems, being there necessarily zero. Then follows the designation of the same properties after w_1 is deleted.

(1)	$c2c2$	VI *J*.
(2)	$c1c2$	VI *I*.
(3)	$c1c1$	II *B*, *I*, *L*, VI *F*.
(4)	$c0c2$	VI *D*.
(5)	$c0c1$	III *B*, VI *C*.
(6)	$n11c2$	VI *H*.
(7)	$n11n11$	II *C*.
(8)	$n10c2$	VI *G*.
(9)	$n10c1$	II *E*, *N*, VI *E*.
(10)	$n10n11$	II *J*.

* L. c., pp. 377, 378.

† L. c., p. 381.

(11) $n10n10$ II A, D, F, G, H, K, M, II′ A, D, F, G, H, K, M.
(12) $n02c2$ V C.
(13) $n01c2$ V D, VIII B, C.
(14) $n01n11$ VII B, D.
(15) $n01n10$ IV A, VII A, C, X B, D.
(16) $n00c2$ V A, A', B, B', VI B, VIII A, A'.
(17) $n00c1$ III A, VI A.
(18) $n00n10$ IX A, B, X A, C.

Consider now the different classes.

Class 3. II L goes into VI F by the transformation

$$\begin{cases} w_3' = w_3 + \tau_1 - \tau_2 - w_2 \\ \tau_1' = \tau_1 + w_1 \\ \tau_2' = \tau_2 - \tau_1 + w_2 . \end{cases}$$

Class 5. In III B, j can be made zero or $2\sqrt{-1}$ by the transformations given for the similar cases for III A in Class 17 below, except that it is not necessary that $1 + \left(\frac{x}{y}\right)^2 = \frac{1}{y}$. If j equals zero, III B will go into VI C by the transformation

$$\begin{cases} w_3' = \frac{w_3}{2} - \frac{\tau_2}{2}, \\ \tau_1' = \frac{\tau_1}{2} + \frac{w_2}{2}, \\ \tau_2' = \frac{\tau_2}{2} + \frac{w_3}{2}. \end{cases}$$

Class 9. If $j \neq 0$, II N goes into VI E by

$$\begin{cases} w_3' = \frac{1}{j} w_3 - \frac{1}{j^2}\tau_2 - \frac{1}{j^3} w_2, \\ \tau_1' = \frac{1}{j^3}\tau_1 + \frac{1}{j^4} w_1, \\ \tau_2' = \frac{1}{j^2}\tau_2 + \frac{1}{j^3} w_2 . \end{cases}$$

Class 11. II′ F goes into II F by $w_3' = -w_3 + \tau_2$,
II′ G goes into II G by $w_3' = -w_3 + \tau_2$.

II′ M if $j \neq 0$ goes into II M by

$$\begin{cases} w_3' = w_3 + \dfrac{1}{j}\tau_2, \\ \tau_1' = \tau_1 \\ \tau_2' = \tau_2 - \dfrac{1}{j} w_2. \end{cases}$$

II′ M if $j = 0$ goes into II′ H by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 - 2w_1, \\ \tau_2' = \tau_2 - w_2. \end{cases}$$

II G goes into II K by

$$\begin{cases} w_3' = -\tfrac{1}{2} w_3 + \tfrac{1}{4}\tau_2, \\ \tau_1' = \quad \tfrac{1}{16}\tau_1 + \tfrac{1}{8} w_1, \\ \tau_2' = -\tfrac{1}{4}\tau_2. \end{cases}$$

II D goes into II A by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 + \dfrac{2}{1-d} w_1, \\ \tau_2' = \tau_2 + \dfrac{1+d}{1-d} w_2. \end{cases}$$

II′ D goes into II′ A by

$$\begin{cases} w_3' = -w_3, \\ \tau_1' = \tau_1 - 2w_1, \\ \tau_2' = -\tau_2 + w_2. \end{cases}$$

II′ K goes into II K by

$$\begin{cases} w_3' = w_3 - \tau_2, \\ \tau_1' = \tau_1, \\ \tau_2' = -\tau_2 - w_2. \end{cases}$$

II M if $j = 0$ goes into II H by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 + \dfrac{2}{1-d} w_1, \\ \tau_2' = \tau_2 + \dfrac{1+d}{1-d} w_2. \end{cases}$$

II M if $j = \dfrac{1-d}{2}$ goes into II F by

$$\begin{cases} w_3' = \dfrac{w_3}{2} - \dfrac{d}{1-d}\tau_2, \\ \tau_1' = \dfrac{1-d}{2}\tau_1 + w_1, \\ \tau_2' = \tau_2 + \dfrac{1+d}{1-d} w_2. \end{cases}$$

II M if $j \neq 0, \frac{1-d}{2}$, goes into II K by

$$\begin{cases} w_3' = \dfrac{(1-d-2j)}{j^2(1-d)} w_3, \\ \tau_1' = \dfrac{(1-d-2j)^2}{j^5(1-d)^2} \tau_1 + \dfrac{2(1-d-2j)^2}{j^5(1-d)^3} w_1, \\ \tau_2' = \dfrac{(1-d-2j)}{j^3(1-d)} \tau_2 + \dfrac{(1-d-2j)(1+d)}{j^3(1-d)^2} w_2. \end{cases}$$

In II K, d can be made zero by

$$\begin{cases} w_3' = (1-d)^2 w_3 - d(1-d)^2 \tau_2, \\ \tau_1' = (1-d)^6 \tau_1, \\ \tau_2' = (1-d)^3 \tau_2 + d(1-d)^3 w_2. \end{cases}$$

Class 13. VIII B goes into V D by

$$\begin{cases} w_3' = w_3 + \tau_2, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 - w_2. \end{cases}$$

Class 14. VII B goes into VII D by

$$\begin{cases} w_3' = w_3 + \frac{1}{2} w_2, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 - \frac{1}{2} w_2. \end{cases}$$

Class 15. If $f \neq -1$ VII A goes into VII C by

$$\begin{cases} w_3' = w_3 + \dfrac{1}{2(f+1)} w_2, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 - \dfrac{1}{2(f+1)} w_2, \end{cases}$$

If $) \neq -1$ VII C goes into IV A by

$$\begin{cases} w_3' = \dfrac{f^{\frac{1}{2}}}{1+f} w_3 + \dfrac{\tau_2}{f^{\frac{1}{2}}(1+f)}, \\ \tau_1' = -\dfrac{\tau_1}{f^{\frac{1}{2}}(1+f)}, \\ \tau_2' = -\dfrac{\tau_2}{1+f} + \dfrac{w_3}{1+} \;. \end{cases}$$

Class 16. V B' goes into V B by

$$\begin{cases} w_3' = w_3 + 2\tau_1 + 2\tau_2, \\ \tau_1' = \tau_2 - w_2, \\ \tau_2' = \tau_1 - w_2. \end{cases}$$

V A' goes into V A by interchanging the τ's.
VIII A' goes into VIII A by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 - \tau_2, \\ \tau_2' = -\tau_2. \end{cases}$$

V B goes into V A by

$$\begin{cases} w_3' = w_3 - \dfrac{2}{1-g}\tau_1, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 + \dfrac{1+g}{1-g}w_2. \end{cases}$$

VIII A, if $g \neq 1$, goes into V A by

$$\begin{cases} w_3' = w_3 - \dfrac{4}{(1-g)^2}w_2 + \dfrac{2}{1-g}\tau_2, \\ \tau_1' = \tau_1 - \dfrac{1+g}{1-g}w_2, \\ \tau_2' = \tau_2 - \dfrac{1}{1-g}w_2 - \dfrac{1}{1+g}\tau_1. \end{cases}$$

In V A, g can be changed to $\dfrac{1}{g}$ by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_2, \\ \tau_2' = \dfrac{1}{g}\tau_1. \end{cases}$$

Class 17. In III A, if $j \neq 0, \pm 2\sqrt{-1}$, it can be made zero by

$$\begin{cases} w_3' = -\dfrac{x^2+2y^2}{\sqrt{x^2+4y^2}}w_3 + \dfrac{xy}{\sqrt{x^2+4y^2}}\tau_2, \\ \tau_1' = -\dfrac{(x^2+y^2)(x^2+2y^2)}{y\sqrt{x^2+4y^2}}\tau_1 - \dfrac{x(x^2+y^2)}{\sqrt{x^2+4y^2}}w_2, \\ \tau_2' = \quad xw_3 + y\tau_2. \end{cases}$$

where $(1)\quad j+\left(\frac{x}{y}\right)^3+3\left(\frac{x}{y}\right)=0\,,$

and $(2)\quad 1+\left(\frac{x}{y}\right)^2=\frac{1}{y}\,.$

Infinite values of the parameters being excluded, $j \neq \infty$, so $\frac{x}{y} \neq \infty$. Therefore from (2) $y \neq 0$. Since $j \neq 0$. $\frac{x}{y} \neq 0$, hence $x \neq 0$. From (2), y is finite unless $\frac{x}{y} = \pm\sqrt{-1}$, in which case $j = \pm\, 2\sqrt{-1}$, which is contrary to hypothesis. y being finite, x is, for $\frac{x}{y}$ is. Since $j \neq \pm\, 2\sqrt{-1}$, $\frac{x}{y} \neq \pm\, 2\sqrt{-1}$, $\pm\sqrt{-1}$ whence $x^2+y^2 \neq 0$, $x^2+4y^2 \neq 0$. These suffice to make the determinant of the transformation finite and different from zero.

III A if $j=0$ goes into VI A by

$$\begin{cases} w_3' = \dfrac{w_3}{2} - \dfrac{\tau_2}{2}\,, \\[2mm] \tau_1' = \dfrac{\tau_1}{2} + \dfrac{w_2}{2}\,, \\[2mm] \tau_2' = \dfrac{\tau_2}{2} + \dfrac{w_3}{2}\,. \end{cases}$$

In III A, if $j = -\,2\sqrt{-1}$, it can be made $+\,2\sqrt{-1}$ by

$$\begin{cases} w_3' = -\,w_3, \\ \tau_1' = -\,\tau_1, \\ \tau_2' = \tau_2. \end{cases}$$

Class 18. In IX A and B f can be reduced to -2 by

$$\begin{cases} w_3' = \dfrac{f^2+4f+2}{2(f+1)(f+3)}\,w_3 + \dfrac{f+2}{2(f+1)(f+3)}\,\tau_2, \\[2mm] \tau_1' = \dfrac{-h(f+2)}{4(f+1)^2(f+3)^2}\,w_1 + \dfrac{f+2}{(f+1)(f+3)}\,w_2 - \dfrac{1}{(f+1)(f+3)}\,\tau_1, \\[2mm] \tau_2' = \dfrac{-h(f+2)}{4(f+1)(f+3)}\,w_2 + \dfrac{f+2}{(f+1)(f+3)}\,w_3 - \dfrac{1}{(f+1)(f+3)}\,\tau_2. \end{cases}$$

X A, if $m \neq 3, -1$, goes into X C by

$$\begin{cases} w_3' = & w_3 + \dfrac{m-1}{2}\tau_2, \\ \tau_1' = - & \dfrac{(m^2-2m-3)(m-1)(m^2-2m+5)}{32} w_2 - \dfrac{m^2-2m-3}{2}\tau_1 + \dfrac{m-1}{2} w_1, \\ \tau_2' = & \dfrac{(m^2-2m+3)(m-1)}{4} w_3 + \tau_1 + \dfrac{m^2-2m+9}{4}\tau_2. \end{cases}$$

The remaining forms are inequivalent, and the parameters can be reduced no further. The proofs of these facts are not difficult, except in the following cases: To prove that X B is distinct from X D. To prove that X A with $m = 3$ and $m = -1$ cannot go into X C. To prove that in X C, m cannot be reduced. To prove that IX A and B are distinct from X. These will be considered in another section.

We have, then, for the linearly independent systems, VI J, VI I, II B, II I, VI F, VI D, III B (with $j = 2\sqrt{-1}$), VI C, VI H, II C, VI G, II E, II N (with $j = 0$), VI E, II J, II A, II F, II H, II K (with $d = 0$), II′ A, II′ H, V C, V D, VIII C, VII D, IV A, VII A (with $f = -1$), VII C (with $f = -1$), X B, X D, V A, VI B, VIII A (with $g = 1$), III A (with $j = 2\sqrt{-1}$), VI A, IX A (with $f = -2$), IX B (with $f = -2$,) X A (with $m = 3$), X A (with $m = -1$), X C.

In II H $d^2 \neq 1$, thus omitting $d = 1$ and $d = -1$. These are precisely II I and II J, which can, therefore, be omitted if we remove the restriction on d in II H. In II A, also, the cases $d = 1$ and $d = -1$ are excluded, which are II B and II C respectively. These can, therefore, in similar manner, be omitted. In V A, $g = 1$ and $g = -1$ are omitted, which are respectively VI D and V C. These will accordingly be left out. In X C, $m \neq 1$, $1 \pm 2\sqrt{-1}$. The second of these is X D, which will be dropped.

The different forms are given in the following table, the letters having the signification given in the general form on p. 380. All forms are linearly independent, except that No. 24 with any value of g is equivalent to the same form with g having the reciprocal value.

	a	b	c	d	e	f	g	h	i	j
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	0	1
4	0	0	1	1	0	1	1	0	1	$2\sqrt{-1}$
5	0	0	0	0	0	1	1	0	0	1
6	0	0	0	0	1	0	0	0	0	0
7	0	0	0	0	1	0	0	1	0	0
8	0	0	1	1	1	0	0	0	0	0
9	0	0	1	1	1	0	0	1	0	0
10	0	0	0	0	1	0	0	0	0	1
11	0	0	1	d	0	0	0	0	0	0
12	0	0	1	0	0	0	0	0	0	1
13	0	0	1	d	0	0	0	1	0	0
14	0	0	1	0	0	0	0	1	0	1
15	0	0	0	1	0	0	0	0	0	0
16	0	0	0	1	0	0	0	1	0	0
17	0	0	0	0	0	1	-1	1	0	0
18	1	0	0	0	0	0	0	1	0	0
19	1	0	1	-1	0	0	0	0	0	0
20	0	0	1	-1	0	1	-1	0	1	0
21	1	0	1	-1	0	-1	1	1	-1	0
22	1	0	1	-1	0	-1	1	0	-1	0
23	1	0	$1 \mp \sqrt{-1}$	$1 \pm \sqrt{-1}$	0	$\mp \sqrt{-1}$	$\pm \sqrt{-1}$	1	1	$\mp 2\sqrt{-1}$
24	0	0	0	0	0	1	g	0	0	0
25	0	0	0	0	1	1	1	0	0	0
26	1	0	0	0	0	1	1	1	0	0
27	0	0	1	1	1	1	1	0	1	$2\sqrt{-1}$
28	0	0	0	0	1	1	1	0	0	1
29	1	0	1	1	0	-2	-2	1	0	0
30	1	0	1	1	0	-2	-2	0	0	0
31	1	0	2	0	0	-3	-1	1	-1	-2
32	1	0	0	2	0	-1	-3	1	-1	2
33	1	0	$\frac{m+1}{m-1}$	$\frac{m-3}{m-1}$	0	$-\frac{m^2+3}{4}$	$-\frac{m^2-4m+7}{4}$	0	$-\frac{(m-1)^2}{4}$	$-(m-1)$

$m \neq 1$

If reciprocal systems are considered equivalent, Nos. 15, 16, 23 with one sign of the radical and 31 may be omitted, being reciprocals respectively of 11, 13, 23 with the other sign of the radical and 32. 11 and 13 are the same as their respective reciprocals with d replaced by $\frac{1}{d}$, hence in those cases d can be restricted to $|d| \nless 1$ and $d = 0$. 33 is the same as its reciprocal with m replaced by $(2 - m)$, hence m can be restricted to the cases when the real part of m is not less than unity.

§4.

We return now to the proofs of inequivalence mentioned on p. 393 as being difficult. The first three are especially so, and will be considered together as follows: The two tables

	w_1	w_2	w_3	τ_1	τ_2
w_1	0	0	0	0	0
w_2	0	0	w_1	0	$\frac{2m}{m-1} w_1$
w_3	0	w_1	w_2	w_1	$\frac{m+1}{m-1} \tau_1 + w_2$
τ_1	0	0	$-w_1$	0	$-\frac{m^2+3}{4} w_1$
τ_2	0	$-\frac{2(m-2)}{m-1} w_1$	$\frac{m-3}{m-1} \tau_1 - w_2$	$-\frac{m^2-4m+7}{4} w_1$	$hw_1 - (m-1)\tau_1 - \frac{(m-1)^2}{4} w_2$

$m \neq 1$

	w_1'	w_2'	w_3'	τ_1'	τ_2'
w_1'	0	0	0	0	0
w_2'	0	0	w_1'	0	$\frac{2m'}{m'-1}w_1'$
w_3'	0	w_1'	w_2'	w_1'	$\frac{m'+1}{m'-1}\tau_1' + w_2'$
τ_1'	0	0	$-w_1'$	0	$-\frac{m'^2+3}{4}w_1'$
τ_2'	0	$\frac{-2(m'-2)}{m'-1}w_1'$	$\frac{m'-3}{m'-1}\tau_1' - w_2'$	$-\frac{m'^2-4m'+7}{4}w_1'$	$h'w_1' - (m'-1)\tau_1' - \frac{(m'-1)^2}{4}w_2'$

$m' \neq 1$

represent any cases of X. Suppose these two are equivalent. The general linear transformation is

$$
\begin{aligned}
w_3' &= x_1 w_1 + x_2 w_2 + x_3 w_3 + x_4 \tau_1 + x_5 \tau_2,\\
\tau_1' &= y_1 w_1 + y_2 w_2 + y_3 w_3 + y_4 \tau_1 + y_5 \tau_2,\\
\tau_2' &= z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 \tau_1 + z_5 \tau_2.
\end{aligned}
$$

From the first,

$$
\begin{aligned}
w_2' = \left(x_5^2 h + 2x_2x_3 + \frac{4x_2x_5}{m-1} - x_4x_5\,\frac{m^2-2m+5}{2}\right)w_1 \\
+ \left(x_3 - \frac{m-1}{2}x_5\right)\left(x_3 + \frac{m-1}{2}x_5\right)w_2 + 2x_5\left(x_3 - \frac{m-1}{2}x_5\right)\tau_1,
\end{aligned}
$$

$$
w_1' = \left(x_3 - \frac{m-1}{2}x_5\right)^2\left(x_3 + \frac{m^2-2m+3}{m-1}x_5\right)w_1.
$$

Obtaining $w_3'\tau_2'$ and $\tau_2'w_3'$ by multiplication, also from the second table (expressing the latter in terms of w_1, w_2, w_3, τ_1, τ_2), adding and subtracting, and

comparing the coefficients of like units, we find that y_3 and y_5 are both zero, also

$$\text{(A)}\quad x_2z_3 + \frac{2}{m-1}x_2z_5 + x_3z_2 - x_4z_5\frac{m^2-2m+5}{4} + x_5z_2\frac{2}{m-1} - x_5z_4\frac{m^2-2m+5}{4} + hx_5z_5 = y_1,$$

$$\text{(B)}\quad x_3z_3 - x_5z_5\frac{(m-1)^2}{4} = y_2,$$

$$\text{(C)}\quad x_3z_5 + x_5z_3 - x_5z_5(m-1) = y_4,$$

$$\text{(D)}\quad 2x_2z_5 + x_3z_4 - x_4z_3 - x_4z_5\frac{m-1}{2} - 2x_5z_2 + x_5z_4\frac{m-1}{2} = \frac{2}{m'-1}y_1 + x_5^2h + 2x_2x_3 + \frac{4x_2x_5}{m-1} - x_4x_5\frac{m^2-2m+5}{2},$$

$$\text{(E)}\quad x_3z_5 - x_5z_3 = \frac{2}{m'-1}y_2 + \left(x_3 - \frac{m-1}{2}x_5\right)\left(x_3 + \frac{m-1}{2}x_5\right),$$

$$\text{(F)}\quad \frac{2x_3z_5}{m-1} - \frac{2x_5z_3}{m-1} = \frac{2}{m'-1}y_4 + 2x_5\left(x_3 - \frac{m-1}{2}x_5\right).$$

Obtaining equations in like manner from $w_2'\tau_2'$ and $\tau_2'w_2'$, subtracting and dividing by $\left(x_3 - \frac{m-1}{2}x_5\right)$, we have

$$\text{(G)}\quad x_3z_5 - x_5z_3 = \left(x_3 - \frac{m-1}{2}x_5\right)\left(x_3 + x_5\frac{m^2-2m+3}{m-1}\right).$$

The division is possible, for $\left(x_3 - \frac{m-1}{2}x_5\right)$ does not equal zero, else w_1' does.

Similarly, from $\tau_1'\tau_2'$ and $\tau_2'\tau_1'$, we obtain

$$\text{(H)}\quad 2y_2z_5 - y_4z_3 - y_4z_5\frac{m-1}{2} = -\frac{m'-1}{2}\left(x_3 - \frac{m-1}{2}x_5\right)^2\left(x_3 + x_5\frac{m^2-2m+3}{m-1}\right).$$

From $\tau_2'^2$ the τ_1 terms and the w_1 terms give, respectively,

$$\text{(I)}\quad 2z_5\left(z_3 - z_5\frac{m-1}{2}\right) = -\frac{(m'-1)^2}{2}x_5\left(x_3 - x_5\frac{m-1}{2}\right) - (m'-1)y_4,$$

$$\text{(J)}\quad z_5^2h + 2z_2z_3 + \frac{4z_2z_5}{m-1} - z_4z_5\frac{m^2-2m+5}{2} = -(m'-1)y_1 + h'\left(x_3 - \frac{m-1}{2}x_5\right)^2\left(x_3 + x_5\frac{m^2-2m+3}{m-1}\right) - \frac{(m'-1)^2}{4}\left(hx_5^2 + 2x_2x_3 + \frac{4x_2x_5}{m-1} - x_4x_5\frac{m^2-2m+5}{2}\right).$$

Substituting from (G) in (E) and (F), we obtain

(K) $y_2 = \dfrac{m'-1}{4(m-1)}\left(x_3 - \dfrac{m-1}{2}x_5\right)x_5(m^2-2m+5)$,

(L) $y_4 = \dfrac{m'-1}{m-1}\left(x_3 - x_5\dfrac{m-1}{2}\right)\left(x_3 + x_5\dfrac{2}{m-1}\right)$.

Substitute from (B) and (C) in (H), factor, use (G), and divide by $\left(x_3 - \dfrac{m-1}{2}x_5\right)\left(x_3 + x_5\dfrac{m^2-2m+3}{m-1}\right)$ (which cannot equal zero, else w_1' does), and there results

(M) $z_3 - z_5\dfrac{m-1}{2} = -\dfrac{m'-1}{2}\left(x_3 - x_5\dfrac{m-1}{2}\right)$.

Substituting from (L) and (M) in (I), and dividing by $(m'-1)\left(x_3 - x_5\dfrac{m-1}{2}\right)$,

(N) $z_5 = \dfrac{m'-1}{m-1}\left(x_3 + x_5\dfrac{m^2-2m+5}{2(m-1)}\right)$.

Substituting in (M) from (N),

(O) $z_3 = \dfrac{(m'-1)(m^2-2m+3)}{2(m-1)}x_5$.

Substitute from (N) and (O) in (G), divide by

$\left(x_3 - \dfrac{m-1}{2}x_5\right)\left(x_3 + x_5\dfrac{m^2-2m+3}{m-1}\right)$, and we obtain

(P) $\dfrac{m'-1}{m-1} = 1$ or $m' = m$.

Hence, m cannot be changed. Replace, therefore, m' by m in the equations and see if h can be changed in the cases X A, $m=3, -1$, and X B. Since, in the latter $m = 1 \pm 2\sqrt{-1}$, these can all three be combined into the case $(m^2-2m+5)(m^2-2m-3)=0$. Let $h=0$, $h'=1$.

Substituting from (K), (L), (N) and (O) in (A),

$$(Q)\left\{\begin{aligned} y_1 = x_2x_5\frac{m^2-2m+3}{2} &+ x_2x_5\frac{m^2-2m+5}{(m-1)^2} + \frac{2x_2x_3}{m-1} + x_3z_2 \\ &- x_4x_3\frac{m^2-2m+5}{4} - x_4x_5\frac{(m^2-2m+5)^2}{8(m-1)} \\ &+ \frac{2x_5z_2}{m-1} - x_5z_4\frac{m^2-2m+5}{4}. \end{aligned}\right.$$

Substitute from (Q) in (D), using (N) and (O), and we have

$$(\mathrm{R})\left\{\begin{aligned} &-\frac{4x_2x_3}{(m-1)^2}-x_2x_5\frac{4(m^2-2m+3)}{(m-1)^3}+x_3z_4+2x_4x_5\frac{m^2-2m+3}{(m-1)^2} \\ &\quad+\frac{2x_3x_2}{m-1}-x_5z_2\frac{2(m^2-2m+3)}{(m-1)^2}+x_5z_4\frac{m^2-2m+3}{m-1}-\frac{2x_3z_2}{m-1}=0. \end{aligned}\right.$$

Substituting in (J) from (O), (N) and (Q),

$$(\mathrm{S})\left\{\begin{aligned} &x_5z_2\frac{(m^2-2m+5)(m^2-2m+3)}{(m-1)^2}+x_3z_2\frac{m^2-2m+5}{m-1}-x_3z_4\frac{m^2-2m+5}{2} \\ &-x_5z_4\frac{(m^2-2m+5)(m^2-2m+3)}{2(m-1)}+x_2x_5\frac{(m^2-2m+3)(m^2-2m+5)}{2(m-1)} \\ &+x_2x_3\frac{m^2-2m+5}{2}-x_4x_5\frac{(m^2-2m+5)(m^2-2m+3)}{4} \\ &-x_4x_3\frac{(m^2-2m+5)(m-1)}{4} \\ &=\left(x_3-\frac{m-1}{2}x_5\right)^2\left(x_3+x_5\frac{m^2-2m+3}{m-1}\right). \end{aligned}\right.$$

Multiply (R) by $\frac{m^2-2m+5}{2}$ and add to (S) and we get, remembering that by hypothesis $(m^2-2m+5)(m^2-2m-3)=0$,

$$(T)\qquad \left(x_3-\frac{m-1}{2}x_5\right)^2\left(x_3+x_5\frac{m^2-2m+3}{m-1}\right)=0,$$

whence $w_1'=0$, which is impossible. Hence the reduction cannot be made.

Consider the fourth case of p. 393 IX A and B, since f can be reduced to -2, are both included in the first of the following tables, the second representing X:

	w_1	w_2	w_3	τ_1	τ_2
w_1	0	0	0	0	0
w_2	0	0	w_1	0	$2w_1$
w_3	0	w_1	w_2	w_1	$w_2+\tau_1$
τ_1	0	0	$-w_1$	0	$-2w_1$
τ_2	0	$-2w_1$	$-w_2+\tau_1$	$-2w_1$	hw_1

	w_1'	w_2'	w_3'	τ_1'	τ_2'
w_1'	0	0	0	0	0
w_2'	0	0	w_1'	0	$\frac{2m}{m-1}w_1'$
w_3'	0	w_1'	w_2'	w_1'	$\frac{m+1}{m-1}\tau_1'+w_2'$
τ_1'	0	0	$-w_1'$	0	$-\frac{m^2+3}{4}w_1'$
τ_2'	0	$\frac{-2(m-2)}{m-1}w_1'$	$\frac{m-3}{m-1}\tau_1'$ $-w_2'$	$-\frac{m^2-4m+7}{4}w_1'$	$h'w_1'-(m-1)\tau_1'$ $-\frac{(m-1)^2}{4}w_2'$

Supposing these to be equivalent, we have

$$
\begin{aligned}
w_3' &= x_1 w_1 + x_2 w_2 + x_3 w_3 + x_4 \tau_1 + x_5 \tau_2 , \\
\tau_1' &= y_1 w_1 + y_2 w_2 + y_3 w_3 + y_4 \tau_1 + y_5 \tau_2 , \\
\tau_2' &= z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 \tau_1 + z_5 \tau_2 ,
\end{aligned}
$$

whence

$$
\begin{aligned}
w_2' &= x_3^2 w_2 + (h x_5^2 + 2 x_2 x_3 - 4 x_4 x_5)\, w_1 + 2 x_3 x_5 \tau_1 , \\
w_1' &= x_3 (x_3^2 - 4 x_5^2)\, w_1 .
\end{aligned}
$$

The products $\tau_1' w_2'$ and $w_2' \tau_1'$ (remembering that w_1', hence $x_3(x_3^2 - 4x_5^2)$, cannot equal zero), show that y_3 and y_5 are both zero. $w_3'\tau_1'$ and $\tau_1' w_3'$ give

(A) $$x_3 y_2 - 2 x_5 y_4 = 0 .$$

The w_2 terms in $w_3'\tau_2'$ and $\tau_2' w_3'$ yield

(B) $$x_3 z_3 = y_2 .$$

$w_2'\tau_2'$ and $\tau_2' w_2'$, after dividing through by $x_3(x_3 - 2x_5)$ and $x_3(x_3 + 2x_5)$ respectively, give

(C) $$z_3 + 2 z_5 = \frac{2m}{m-1}(x_3 + 2x_5) .$$

(D) $$z_3 - 2 z_5 = \frac{-2(m-2)}{m-1}(x_3 - 2x_5) ,$$

whence

(E) $$z_3 = \frac{2x_3}{m-1} + 4x_5 .$$

$\tau_1'\tau_2'$ and $\tau_2'\tau_1'$ yield respectively,

(F) $$(y_2 - y_4)(z_3 + 2z_5) = -\frac{m^2+3}{4}(x_3^2 - 4x_5^2)\, x_3 ,$$

(G) $$(y_2 + y_4)(z_3 - 2z_5) = -\frac{m^2 - 4m + 7}{4}(x_3 - 4x_5^2)\, x_3 .$$

Now it is impossible that m equal zero, for then, from (C), $z_3 + 2z_5 = 0$, whence, from (F), $x_3(x_3^2 - 4x_5^2) = 0$, which is impossible. Similarly, m does not equal two. Hence, we can divide (F) and (G) by (C) and (D) respectively, obtaining

$$
\begin{aligned}
y_2 - y_4 &= -\frac{(m^2+3)(m-1)}{8m}(x_3 - 2x_5)\, x_3 , \\
y_2 + y_4 &= \frac{(m^2 - 4m + 7)(m-1)}{8(m-2)}(x_3 + x2_5)\, x_3 ,
\end{aligned}
$$

from which

(H) $$y_2 = \frac{x_3(m-1)}{8m(m-2)}\left[-x_3(m^2-2m-3)+2x_5(m-1)(m^2-2m+3)\right],$$

(I) $$y_4 = \frac{x_3(m-1)}{8m(m-2)}\left[\ x_3(m-1)(m^2-2m+3)-2x_5(m^2-2m-3)\right].$$

Substituting from (H) and (I) in (A), there results

$$(m^2-2m-3)(x_3^2-4x_5^2)=0,$$

whence $m^2-2m-3=0$, giving $m=3$ or -1. In either of these cases substitute from (E) and (H) in (B), obtaining respectively

$$x_3+2x_5=0 \text{ or } x_3-2x_5=0,$$

either of which is impossible.

Hence, IX is distinct from X.

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from which

$$\text{(H)} \qquad y_2 = \frac{x_3(m-1)}{8m(m-2)}\left[-x_3(m^2-2m-3)+2x_5(m-1)(m^2-2m+3)\right],$$

$$\text{(I)} \qquad y_4 = \frac{x_3(m-1)}{8m(m-2)}\left[\ x_3(m-1)(m^2-2m+3)-2x_5(m^2-2m-3)\right].$$

Substituting from (H) and (I) in (A), there results

$$(m^2-2m-3)(x_3^2-4x_5^2)=0,$$

whence $m^2-2m-3=0$, giving $m=3$ or -1. In either of these cases substitute from (E) and (H) in (B), obtaining respectively

$$x_3+2x_5=0 \quad \text{or} \quad x_3-2x_5=0,$$

either of which is impossible.

Hence, IX is distinct from X.

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